

Generalized SU(2) Proca Theory

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Following previous works on generalized Abelian Proca theory, also called vector galileon, we investigate the massive extension of a SU(2) gauge theory, i.e. the generalized SU(2) Proca model, which could be dubbed non-Abelian vector galileon. This particular symmetry group permits fruitful applications in cosmology such as inflation driven by gauge fields. Our approach consists in building in an exhaustive way all the Lagrangians containing up to six contracted Lorentz indices. For this purpose, and after identifying by group theoretical considerations all the independent Lagrangians which can be written at these orders, we consider the only linear combinations propagating three degrees of freedom and having healthy dynamics for their longitudinal mode, i.e. whose pure Stückelberg contribution turns into the SU(2) multi-galileon dynamics. Finally, and after having considered the curved space-time expansion of these Lagrangians, we discuss the form of the theory at all subsequent orders.

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I. INTRODUCTION

In the search of well motivated theories that describe the primordial universe, several attempts have been made to obtain inflationary descriptions from particle physics (the Standard Model, Supersymmetry, Grand Unified Theories, etc., see e.g. Refs. [1–5]), or from quantum theories of gravity such as Supergravity, String Theory, and Loop Quantum Gravity, see e.g. Refs. [6–10]. This top-down approach has been very fruitful, providing new ways to understand the structure of the high energy theories necessary to reproduce the observable properties of the Universe, ranging from the Cosmic Microwave Background Radiation (CMB) to the Large-Scale Structure (LSS). However, little is known from the observational point of view for many of these theories (those whose characteristic energy scale is much higher than the electroweak one), the CMB and LSS being at present the only situations in which they would have had observable consequences and would thus leave testable signatures. Since the power of the current and proposed accelerators is not going to increase as much as would be needed to directly test these theories in any foreseeable future, we need to devise another approach to the fundamental theory that describes nature.

Such an approach already exists, and boils down to the question on whether there is any choice in formulating the fundamental theory. This bottom-up approach consists in finding out an action completely

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free of pathologies, the first of them being the Ostrogradski instability [11] (the Hamiltonian could be unbounded from below), and satisfying a given set of assumptions, e.g., symmetry requirements. One then needs to define the material content of the universe (scalar fields, vector fields, ...), although in principle, the construction itself and the stability requirements constrain some content and allow others so that the material content is, once the conditions are applied, somehow redefined. This very ambitious program is just beginning to be implemented and interesting works have been carried out in which the extra material content (on top of gravity) is composed by one or many scalar fields. It was Horndeski [12] who found for the first time the most general action for a scalar field and gravity that produces second-order equations of motion. In general, if the Lagrangian is nondegenerate, having equations of motion of second-order at most is a necessary requirement to avoid the Ostrogradsky instability [13, 14]. By pursuing this goal, an action is found that, however, still requires a Hamiltonian analysis in order to guarantee that the instability is not present.

Horndeski's construction was rediscovered in the context of what is nowadays called Galileons [15]. The Galileons are the scalar fields whose action, in flat spacetime, leads to equations of motion that involve only second-order derivatives. The idea has been extended finding out the so called Generalized Galileons by allowing for lower-order derivatives in the equations of motion [16, 17]. The background space-time geometry where these Generalized Galileons live can be promoted to a curved one by promoting the ordinary derivatives to covariant ones and adding some counterterms that involve non-minimal couplings to the curvature [18, 19]. The latter guarantees the equations of motion for both geometry and matter are still second-order so that the Galileons, both Generalized and Covariantized, are found. This procedure is equivalent to that proposed by Horndeski for one scalar field [20], but loses some interesting terms when more than one scalar field is present [21]. The Galileon approach for scalar fields has found multiple applications in cosmology, ranging from inflation (see eg. [22–32]) to dark energy (see e.g. [33–50]).

The original proposal was based on the requirement of second order equations of motion for all the additional degrees of freedom to gravity, all of them being therefore dynamical so that the system is non degenerate. The generalization to the so-called extended Horndeski theories also include non physical degrees of freedom and thus considers degenerate theories [51–55]. Such a construction is by now well understood and some cosmological applications have also been considered [55–63].

However, scalar fields are not the only possibilities as the matter content of the universe. Horndeski indeed wondered some fourty years ago what the action would be for an Abelian vector field in curved spacetime [64]. Working with curvature is a way to bypass the no-go theorem presented in Ref. [65] which states that the only possible action for an Abelian vector field in flat spacetime that leads to second-order equations of motion is the Maxwell-type one. Relaxing the gauge invariance allows for a non trivial action in flat spacetime generalizing this way the Proca action [66, 67]. The construction of the generalized Proca action has been well investigated and discussed so there is already a consensus about the number and type of terms in the action, even in the covariantized version [68–70]. Moreover, the analogous extended Horndeski theories have been built for a vector field [71, 72] and the corresponding cosmological applications have been explored [67, 73–78].

Some cosmological applications of vector fields had been investigated and interesting scenarios, such as the fF^2 model [79] and the vector curvaton [80, 81], were devised. There is however an obstacle when dealing with vector fields in cosmology: they produce too much anisotropy, both at the background and perturbation levels, well above the observable limits unless one implements some dilution mechanism or if one considers only the temporal component of the vector field (which is however usually non dynamical). In the fF^2 model, the potentially huge anisotropy is addressed by coupling the vector field to a scalar that dominates the energy density of the universe and, therefore, dilutes the anisotropy; in contrast, in the vector curvaton scenario, the anisotropy is diluted by the very rapid oscillations of the vector curvaton around the minimum of its potential. Another dilution mechanism is to consider many randomly oriented vector fields [82] but this requires a large number of them, indeed hundreds, so that it is difficult to justify it from a particle physics point of view. There is, nevertheless, another possibility, the so-called “cosmic triad” [82, 83], situation in which three vector fields orthogonal to each other and of the same norm, can give rise to a rich phenomenology while making the background and perturbations completely isotropic [84]. A couple of very interesting models, gauge-flation [85, 86] and chromo-natural inflation [87], have implemented this idea by embedding it in a non-Abelian framework and exploiting the local isomorphism between the $SO(3)$ and $SU(2)$ groups of transformations. At first sight, the cosmic triad configuration looks very unnatural, but dynamical system studies have shown that it represents an attractor configuration [88]. Unfortunately, although the background dynamics of these two models is successful, their perturbative dynamics makes them incompatible with the latest Planck observations [89, 90]. Despite this failure, such models have shown the applicability that non-Abelian gauge fields can have in cosmological scenarios.

Having in mind the above motivations, the purpose of this paper is to build the first order terms of

the generalized SU(2) Proca theory, and to discuss the general form of the complete theory. For the most part, we focus on those Lagrangians containing up to six contracted Lorentz indices, that we obtain exhaustively. To ensure not to forget some terms, we first construct from group theoretical considerations all possible Lagrangians at these orders, before imposing the standard dynamical condition, i.e. that only three degrees of freedom propagate. Then, after identifying all the Lagrangians which imply the same dynamics, e.g. those related by a conserved current, we verify that the pure Stückelberg part of the Lagrangians is healthy, i.e. implies the SU(2) multigalileon dynamics. To this end, it is useful to derive all the equivalent formulations of the SU(2) adjoint multi-galileon model, which we provide in the Appendix. Then, after having computed the relevant curved space-time extension of our Lagrangians, we conclude on the status of the complete formulation of the theory, i.e. that containing the higher order terms we did not consider in this work.

The layout of this paper is the following. In Section II, the generalized non-Abelian Proca theory is introduced and some technical aspects needed for later sections are laid out; the procedure to build the theory is also described. In Section III, the building blocks of the Lagrangian are systematically obtained. Section IV deals with the right number of propagating degrees of freedom and the consistency of the obtained Lagrangian with the scalar galileon nature of its longitudinal part. The covariantization of the theory is performed in Section V and the final model together with a discussion and comparison with the Abelian case is presented in Section VI. The appendix presents the construction of the multi-galileon scalar Lagrangian in the 3-dimensional representation of SU(2) and its equivalent formulations. Throughout this paper, we have employed the mostly plus signature, i.e. $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$, and set $\hbar = c = 1$.

II. GENERALIZED NON-ABELIAN PROCA THEORY

Our aim is to generalize the non-Abelian Proca theory, described below, to include all possible second-order ghost-free terms propagating only three degrees of freedom. After discussing the general symmetry case, we concentrate on the SU(2) symmetry, which is particularly interesting in a cosmological perspective, as discussed in the introduction, and roughly present the procedure which will be thoroughly explained below.

A. Non-Abelian Proca Theory

Let us first present the nowadays standard non-Abelian Proca theory. Also called massive Yang-Mills models, they have been extensively studied in the past, as e.g. in Refs. [91–95], with a Hamiltonian formulation detailed in Ref. [96, 97]. Our starting point Lagrangian, including the mass term, reads

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu}_a + \frac{1}{2}m^2 A_\mu^a A^\mu_a, \quad (1)$$

with the non-Abelian Faraday tensor given by

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{bc}^a A_\mu^b A_\nu^c, \quad (2)$$

g being the coupling constant and f_{bc}^a the structure constants of the symmetry group under consideration. This can be considered as the limit of a valid particle physics model based on a Higgs condensate whose corresponding degree of freedom is assumed frozen, hence breaking the relevant symmetry [73, 74].

Let us emphasize a technical point at this stage: one could work with the vector field assumed as an operator, namely

$$A_\mu(x) = A_\mu^a(x)T_a, \quad (3)$$

with T_a representing the operators associated with corresponding elements of the underlying group in a given representation. We then have, by definition of the algebra, the commutation relations

$$[T_a, T_b] = if_{ab}^c T_c. \quad (4)$$

Since this work concentrates on the vector fields themselves and not on their action on other fields, it is simpler to restrict attention to the fields themselves, i.e.

$$A_\mu(x) = \{A_\mu^a(x)\}, \quad (5)$$

which are in the Lie algebra of the symmetry group under consideration. These two ways of writing the field operators are of course strictly equivalent, but the latter formalism, with group indices attached to the vectors themselves, merely does not need the introduction of the algebra operators themselves and is thus more appropriate for our purpose.

Any action needs to be a scalar, and this includes not only the Lorentz group, but also any internal symmetry, such as that stemming from the algebra in Eq. (4). If the relevant symmetry were of the local type, and for an infinitesimal transformation, the vectors would transform through

$$\delta A_\mu^a = -\frac{1}{g}\partial_\mu \alpha^a(x) + f_{bc}^a \alpha^b(x) A_\mu^c, \quad (6)$$

which leaves invariant only the kinetic term $F_{\mu\nu}^a F_a^{\mu\nu}$, but of course not even a mass term $A_\mu^a A_a^\mu$, much less any extension such as those we want to consider below. This is merely a restatement of the well-known fact that mass breaks gauge symmetry. We therefore restrict attention to global transformations of the kind

$$\partial_\mu \alpha^a = 0 \quad \implies \quad \delta A_\mu^a = f_{bc}^a \alpha^b A_\mu^c, \quad (7)$$

i.e. we assume the vector field itself transforms as the adjoint representation, with dimension equal to that of the symmetry group itself. It is also profitable, and maybe more enlightening, to look at the effect of a finite local transformation of the group, still described by a set of parameters $\alpha^a(x)$. Under this transformation, the vector field transforms as

$$A_\mu(x) = A_\mu^a(x) T_a \mapsto U[\alpha^a(x)] \left[-\frac{1}{g}\partial_\mu + A_\mu(x) \right] U^{-1}[\alpha^a(x)], \quad (8)$$

where $U[\alpha^a(x)]$ describes the action of the group element labeled by $\alpha^a(x)$. It permits to emphasize that in the case where the symmetry becomes global, i.e. where $\alpha^a(x)$ does not longer depend on the space-time point, the vector field transforms exactly as the adjoint representation of the symmetry group. This is indeed the symmetry assumed for the non-Abelian Proca (massive Yang-Mills) field. In the Abelian case, this transformation is trivial because the action of the group commutes with the vector field, and the transformation in Eq. (8) thus reduces to the identity in the global symmetry case. In the non-Abelian case, however, one needs to specify how the extra indices are to be summed over in order to produce a singlet with respect to this global symmetry transformation. To relate the set of theories under considerations here with the more usual ones in particle physics involving a local symmetry broken by means of a Higgs field, one can envisage our transformation in Eq. (7) as the limit of that in Eq. (6).

With these motivating considerations, we now move on to evaluating the most general theory with a massive vector field transforming according to the adjoint representation of a given global symmetry group.

B. Restricting Attention to the SU(2) Case

In view of the potentially relevant cosmological consequences, we shall from now on restrict attention to the case for which the relevant symmetry group is SU(2), with dimension equal to 3, and therefore consider a vector field also of dimension 3. Since SU(2) is locally isomorphic to SO(3), one can then simply use a vector representation with group indices varying from 1 to 3 in A_μ^a , i.e. we restrict attention to the fundamental representation of SO(3).

The set of SU(2) structure constants is identical to the 3-dimensional Levi-Civita tensor ϵ_{bc}^a whereas the group metric g_{ab} , given by $g_{ab} = -f_{ad}^e f_{bc}^d$, is simply the flat metric $2\delta_{ab}$. The only primitive invariants are ϵ_{abc} and δ_{ab} [98–100], and one can therefore write all possible contractions by merely contracting fields with contravariant indices with all appropriate combinations of those two primitive invariants written with covariant indices. Recall also the further simplification induced by the fact that contractions among structure constants (Levi-Civita symbols in the case at hand) leaving one, two or three free indices will respectively lead to a vanishing result, or terms proportional to δ_{ab} and ϵ_{abc} [101]; it is therefore often unnecessary to use multiple contractions.

As already alluded to earlier, choosing SU(2) is not innocuous as we aim at cosmological applications, in view in particular of implementing inflation driven by gauge fields (see e.g. Refs. [85–90, 102–115]): since its adjoint representation is 3-dimensional, SU(2) permits to generate configurations for which all the three vectors are non vanishing while ensuring isotropy.

C. Generalization

What follows is very similar to the generalized Abelian Proca case as discussed e.g. in Refs. [66–70, 116] (see also Refs. [117–119] for the equivalent curved space-time construction). In brief, we want to construct the most general action generalizing that of Proca for a massive SU(2) vector field, i.e.

$$\mathcal{S}_{\text{Proca}} = \int \mathcal{L}_{\text{Proca}} d^4x = \int \left(-\frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu} + \frac{1}{2} m_A^2 X \right) d^4x, \quad (9)$$

where $X \equiv A_\mu^a A_\mu^a$. To the above action (9), we want to add all possible terms containing not only functions of X but also derivative self-interactions. These terms will have to fulfill some conditions for the corresponding theory to make sense. We first split the vector into a scalar/vector decomposition

$$A_\mu^a = \partial_\mu \pi^a + \bar{A}_\mu^a, \quad (10)$$

where π^a is a scalar multiplet in the **3** representation of SU(2), i.e. the Stückelberg field generalized to the non-Abelian case, and \bar{A}_μ^a is a divergence-free vector ($\partial_\mu \bar{A}^{\mu a} = 0$), containing the curl part of the field, i.e. that for which the Faraday tensor is non vanishing. The conditions one then must impose on the theory in order that it makes (classical) sense are

- a) the equations of motion for all physical degrees of freedom, i.e., for both \bar{A}_μ^a and π^a , and hence A_μ^a and π^a , must be at most second order, thus ensuring stability [11, 13, 14],
- b) the action may contain at most second-order derivative terms in π^a and first-order derivatives for A_μ^a ,
- c) each component of the SU(2) multiplet propagates only three degrees of freedom, the zeroth component being non dynamical.

In what follows, we shall apply these conditions and restrict attention to these theories involving terms with up to six Lorentz indices contracted. From the cosmological perspective, such theories are expected to allow for a richer phenomenology, since this is what happens for the Abelian Proca case [67, 73–78]

D. Procedure

We shall now proceed along the lines of Ref. [68], i.e. we shall build, in Sec. III, a complete basis of linearly independent test Lagrangians describing all possible Lagrangians containing a given number of vector fields and their derivatives; the detailed prescription is explicated in Sec. III A. Next we demand only three degrees of freedom per multiplet component of the vector field, which translates into a condition on the Hessian [66, 68], the latter being defined through

$$\mathcal{H}^{\mu\nu de} = \frac{\partial}{\partial(\partial_0 A_{\mu d})} \frac{\partial}{\partial(\partial_0 A_{\nu e})} \mathcal{L}, \quad (11)$$

for a given Lagrangian \mathcal{L} . This functional over the fields is symmetric under the index exchange $(\mu, d) \leftrightarrow (\nu, e)$.

In order for $\mathcal{H}^{\mu\nu de}$ to have three vanishing eigenvalues, one for each timelike component of the three vectors $A_{\mu d}$, and since all the terms it is built of are a priori independent (up to symmetries), a necessary condition is that we demand $\mathcal{H}^{0\nu de} = 0$; this requirement will be explicitly checked in Sec. IV A for each test Lagrangian.

The above condition is however not sufficient, for it does not exhaust all the constraints and thus does not count the effectively propagating degrees of freedom. For instance, some terms inducing no dynamics for the time components of the vector fields may also yield no dynamics for some other component, or even for the overall vector field. The required analysis is tedious and must be followed step by step [96, 97].

As the final step of the above analysis, we will consider the scalar part associated with those linear combinations of test Lagrangians verifying the Hessian condition. One must check, which is done in Sec. IV D, that they are of two kinds: either they have no dynamics at all, being vanishing or given by a total derivative, or their dynamics is second order in the equations of motion of the scalar field, i.e. they belong to the class of generalized galileons [100, 120–124]. This will provide the most general terms verifying the requirements we demand, formulated in terms of the non-Abelian Faraday tensor, see Sec. IV E.

Before moving on, we would like to mention that even though the procedure discussed above and applied below is allegedly tedious, it however guarantees an exhaustive list of all possible terms at each order, and in particular all those specific to the non-Abelian case. Those might have been obtained by some quicker method, but we prefer to be able to produce all the theoretically acceptable terms rather than constructing a few. In view of possible cosmological applications, there is indeed no way to say which terms will be relevant and which ones will not.

III. CONSTRUCTION OF THE TEST LAGRANGIANS

As anticipated above, our method relies heavily on the construction of a basis of test Lagrangians satisfying the symmetry requirement, on which we shall later apply the Hessian condition. This is the purpose of this section.

A. Description of the Procedure

We shall now proceed to build the complete basis, in the sense of linear algebra, of test Lagrangians, for a given number of fields and their first derivatives. Being linearly independent, we will then be able to write down the most general theory at the given order as a linear combination of these Lagrangians.

In order to construct Lagrangians, i.e. scalars, we need to consider the Lorentz and group indices. The former, spacetime indices, run from 0 to 3 and are denoted by small greek letters, while the latter group indices run from 1 to 3 since we assume the adjoint 3-dimensional SU(2) representation, and are represented by small latin letters from the beginning of the alphabet. We shall first write down all the Lorentz scalar quantities that may be formed with a given number of fields and first derivatives, and then consider all the SU(2) index combinations leading to SU(2) scalars of these Lorentz scalars.

For the sake of simplicity, beginning with the Lorentz sector, we shall dismiss the group indices altogether, keeping in mind however that their presence might spoil some symmetry properties: contractions between symmetric and anti-symmetric (with respect to Lorentz indices only) tensors will not necessarily vanish when group indices are included, as exemplified by the starred equations in the next section. The Lorentz scalars, once formed, will then subsequently be assigned SU(2) indices following simple alphabetical order, leaving as many free SU(2) indices as there are fields in the term, to be then contracted with a relevant pure SU(2) tensor. For instance, a term like $A^\mu A^\nu (\partial_\mu A^\nu)$ will be indexed as $A^{\mu a} A^{\nu b} (\partial_\mu A^{\nu c})$, demanding contraction with a structure constant ϵ_{abc} to form a Lorentz and SU(2) scalar. This procedure can seem rather tedious, and it most definitely is, but it ensures we construct a complete basis.

For simplicity, we shall restrict attention to those Lagrangians containing up to 6 Lorentz indices contracted as they should to form a scalar.

B. Lorentz Sector

An easy way to classify the Lorentz scalars one can form with a given number of 4-vectors consists in using the equivalence, at the Lie-algebraic level, between O(3,1) and SU(2)×SU(2) (see e.g. Ref. [125]). One obtains the following table [98, 126]:

# of vector fields A^μ	1	2	3	4	5	6	7	8
# of Lorentz scalars	0	1	0	4	0	25	0	196

These scalars can be written in terms of the primitive invariants, namely $g_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$. As shown on the table, an odd number of vector fields is impossible, as is obvious from the fact that one cannot form primitive Lorentz invariants with an odd number of indices. For two fields, the only contracting possibility is $g_{\mu\nu}$, while for four free Lorentz indices, the contractions with a term of the form $A^\mu B^\nu C^\rho D^\sigma$ can be performed with any member of the list

$$\left\{ \begin{array}{l} g_{\mu\nu} g_{\rho\sigma}, \\ g_{\mu\rho} g_{\nu\sigma}, \\ g_{\mu\sigma} g_{\nu\rho}, \\ \epsilon_{\mu\nu\rho\sigma}. \end{array} \right. \quad (12)$$

For the case with six free indices of the form $A^\mu B^\nu C^\rho D^\sigma E^\delta F^\epsilon$, one finds the fifteen independent possibilities of combining three metrics, i.e. $g_{\mu\nu}g_{\rho\sigma}g_{\delta\epsilon}$ and the non-equivalent permutations of indices, as well as fifteen combinations of a metric and a Levi-Civita tensor, of which only ten are independent, which we choose to be

$$\left\{ \begin{array}{l} g_{\nu\rho}\epsilon_{\mu\sigma\delta\epsilon}, \\ g_{\nu\sigma}\epsilon_{\mu\rho\delta\epsilon}, \\ g_{\nu\delta}\epsilon_{\mu\rho\sigma\epsilon}, \\ g_{\nu\epsilon}\epsilon_{\mu\rho\sigma\delta}, \\ g_{\rho\sigma}\epsilon_{\mu\nu\delta\epsilon}, \\ g_{\rho\delta}\epsilon_{\mu\nu\sigma\epsilon}, \\ g_{\rho\epsilon}\epsilon_{\mu\nu\sigma\delta}, \\ g_{\sigma\delta}\epsilon_{\mu\nu\rho\epsilon}, \\ g_{\sigma\epsilon}\epsilon_{\mu\nu\rho\delta}, \\ g_{\epsilon\delta}\epsilon_{\mu\nu\rho\sigma}. \end{array} \right. \quad (13)$$

Now, one needs to take into account that when only one vector A^μ and its gradient are plugged into these expressions, some terms are identical and can thus be simplified. The following table sums up the number of independent terms that can be built for a given product of vectors and gradients. Numbers in parenthesis indicate those terms that would vanish if it were not for the group index; in our listings of all available Lagrangians below, we shall point these contractions by means of a star. Given the above discussion, we are sure that all the possible terms have been found, and they are all linearly independent.

$\#(\partial^\mu A^\nu) \backslash \#A^\rho A^\sigma$	0	1	2
1	1 (0)	3 (1)	6 (4)
2	4 (0)	13 (3)	34 (23)
3	9 (2)	52 (22)	

We shall now discuss each case separately.

For a single derivative and no additional field, one gets the simplest combination, namely $(\partial \cdot A)$. With two additional fields, one gets

$$\left\{ \begin{array}{l} (\partial \cdot A) (A \cdot A), \\ [(\partial^\mu A^\nu) A_\mu A_\nu], \\ [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu) A^\rho A^\sigma], \end{array} \right. \quad (*) \quad (14)$$

and with four additional fields, one obtains

$$\left\{ \begin{array}{l} (\partial \cdot A) (A \cdot A) (A \cdot A), \\ [(\partial^\mu A^\nu) A_\mu A_\nu] (A \cdot A), \\ [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu) A^\rho A^\sigma] (A \cdot A), \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\alpha) A^\nu A^\rho A^\sigma A_\alpha], \quad (*) \\ [\epsilon_{\mu\nu\rho\sigma} (\partial^\alpha A^\mu) A^\nu A^\rho A^\sigma A_\alpha], \quad (*) \\ (\partial \cdot A) [\epsilon_{\mu\nu\rho\sigma} A^\mu A^\nu A^\rho A^\sigma]. \quad (*) \end{array} \right. \quad (15)$$

With two derivatives and no additional field, one then finds

$$\left\{ \begin{array}{l} (\partial \cdot A) (\partial \cdot A), \\ [(\partial^\mu A^\nu) (\partial_\mu A_\nu)], \\ [(\partial^\mu A^\nu) (\partial_\nu A_\mu)], \\ [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu) (\partial^\rho A^\sigma)], \end{array} \right. \quad (16)$$

whereas with two additional fields, one finds¹

¹ As an example of the fact that not all reshuffling of indices are independent, let us consider the term $\epsilon_{\mu\nu\rho\sigma} A^\mu A^\nu (\partial^\alpha A^\rho) (\partial_\alpha A^\sigma)$, which could in principle have appeared in the list in Eq. (18). It is indeed not necessary

$$\left\{ \begin{array}{l}
(\partial \cdot A) (\partial \cdot A) (A \cdot A), \\
[(\partial^\mu A^\nu) (\partial_\mu A_\nu)] (A \cdot A), \\
[(\partial^\mu A^\nu) (\partial_\nu A_\mu)] (A \cdot A), \\
[\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu) (\partial^\rho A^\sigma)] (A \cdot A), \\
[(\partial^\mu A^\nu) A_\mu A_\nu] (\partial \cdot A), \\
[\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu) A^\rho A^\sigma] (\partial \cdot A), \quad (*) \\
[A_\mu A_\nu (\partial^\mu A^\alpha) (\partial^\nu A_\alpha)], \\
[A_\mu A_\nu (\partial^\mu A^\alpha) (\partial_\alpha A^\nu)], \\
[A_\mu A_\nu (\partial^\alpha A^\mu) (\partial_\alpha A^\nu)], \\
[\epsilon_{\mu\nu\rho\sigma} A^\mu A^\nu (\partial^\rho A^\alpha) (\partial^\sigma A_\alpha)], \quad (*) \\
[\epsilon_{\mu\nu\rho\sigma} A^\mu A^\nu (\partial^\rho A^\alpha) (\partial_\alpha A^\sigma)], \quad (*) \\
[\epsilon_{\mu\nu\rho\sigma} A^\mu A^\alpha (\partial^\nu A^\rho) (\partial^\sigma A_\alpha)], \\
[\epsilon_{\mu\nu\rho\sigma} A^\mu A^\alpha (\partial^\nu A^\rho) (\partial_\alpha A^\sigma)].
\end{array} \right. \quad (18)$$

Finally, demanding three gradients of the vector field and no vector field itself, one obtains

$$\left\{ \begin{array}{l}
(\partial \cdot A) (\partial \cdot A) (\partial \cdot A), \\
[(\partial^\mu A^\nu) (\partial_\mu A_\nu)] (\partial \cdot A), \\
[(\partial^\mu A^\nu) (\partial_\nu A_\mu)] (\partial \cdot A), \\
[\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\nu) (\partial^\rho A^\sigma)] (\partial \cdot A), \\
[(\partial^\mu A_\nu) (\partial^\nu A_\rho) (\partial^\rho A_\mu)], \\
[(\partial^\mu A_\nu) (\partial^\nu A_\rho) (\partial_\mu A^\rho)], \\
[\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\alpha) (\partial^\nu A_\alpha) (\partial^\rho A^\sigma)], \quad (*) \\
[\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^\alpha) (\partial_\alpha A^\nu) (\partial^\rho A^\sigma)], \\
[\epsilon_{\mu\nu\rho\sigma} (\partial^\alpha A^\mu) (\partial_\alpha A^\nu) (\partial^\rho A^\sigma)]. \quad (*)
\end{array} \right. \quad (19)$$

C. Group Sector

Let us now proceed at the similar procedure but now in the group sector. Since we assumed the vector fields to transform according to the representation of dimension 3 of SU(2), one can safely use known results from representation theory of compact Lie groups. The table below summarizes the different possibilities to obtain a SU(2) singlet as a function of the number of fields belonging to the **3** representation of SU(2) [98, 126]:

# vector fields in the 3 of SU(2)	1	2	3	4	5	6	7
# of SU(2) singlets	0	1	1	3	6	15	36

We reproduce below the procedure explained in Sec. III A, whereby one constructs the necessary products of group metric coefficients δ_{ab} and structure constants ϵ_{abc} . Getting as many independent terms as predicted by the representation theory (table above) ensures completeness of the basis. Similarly to the Lorentz invariance discussed in the previous section, these two tensors are the only primitive invariants of the group [98–100].

To contract with two or three free SU(2) indices, the only possible choices are respectively δ_{ab} and ϵ_{abc} . With four fields, one can make use of the three combinations

$$\left\{ \begin{array}{l}
\delta_{ab}\delta_{cd}, \\
\delta_{ac}\delta_{bd}, \\
\delta_{ad}\delta_{bc},
\end{array} \right. \quad (20)$$

because the property

$$g_{\mu\rho}\epsilon_{\nu\sigma\delta\epsilon} = g_{\nu\rho}\epsilon_{\mu\sigma\delta\epsilon} - g_{\rho\sigma}\epsilon_{\mu\nu\delta\epsilon} + g_{\rho\delta}\epsilon_{\mu\nu\sigma\epsilon} - g_{\rho\epsilon}\epsilon_{\mu\nu\sigma\delta} \quad (17)$$

allows to write it as a linear combination of the terms in Eq. (18).

while five fields demand the following six possibilities, namely

$$\left\{ \begin{array}{l} \delta_{ab}\epsilon_{cde}, \\ \delta_{ac}\epsilon_{bde}, \\ \delta_{ad}\epsilon_{bce}, \\ \delta_{bc}\epsilon_{ade}, \\ \delta_{bd}\epsilon_{ace}, \\ \delta_{cd}\epsilon_{abe}. \end{array} \right. \quad (21)$$

As in Sec. III B, one can devise possible other formulations that apply, but they will always be expressible as linear combinations of the above. For instance, relations between the structure constants such as

$$\epsilon_{ab}{}^e \epsilon_{cde} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}, \quad (22)$$

imply that contracting a four-index term with two structure constants is equivalent to a linear combination of the terms given in Eq. (20).

D. Final Test Lagrangians

Gathering the results and applying the procedure of Sec. III A, we are now in a position to write down our test Lagrangians, scalars under both Lorentz and SU(2) transformations. Some of these terms simplify through contractions, e.g. $\epsilon_{abc}(A^a \cdot A^b)(\partial \cdot A^c) = 0$, and we are left with fewer terms than the naive multiplication of all singlet possibilities of each sector would have otherwise suggested. This is fortunate because the number of terms to be considered a priori is quickly increasing with the number of fields involved, as shown in the table below:

$\# \partial^\mu A^{\nu a} \backslash \# A^{\rho b}$	0	2	4
1	0	3	36
2	4	42	510
3	9	312	

After simplifications, we find two terms (instead of three according to the table) containing a single derivative term and two additional vector fields

$$\left\{ \begin{array}{l} \mathcal{L}_1 = \epsilon_{abc} [(\partial^\mu A^{a\nu}) A_\mu^b A_\nu^c], \\ \mathcal{L}_2 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\nu}) A^{b\rho} A^{c\sigma}], \end{array} \right. \quad (23)$$

and eight with four such fields, namely

$$\left\{ \begin{array}{l} \mathcal{L}_1 = \epsilon_{abc} [(\partial^\mu A^{d\nu}) A_\mu^a A_\nu^b] (A^c \cdot A_d), \\ \mathcal{L}_2 = \epsilon_{abc} [(\partial^\mu A^{a\nu}) A_\mu^d A_\nu^b] (A^c \cdot A_d), \\ \mathcal{L}_3 = \epsilon_{abc} [(\partial^\mu A^{a\nu}) A_\mu^b A_\nu^d] (A^c \cdot A_d), \\ \mathcal{L}_4 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{d\nu}) A^{a\rho} A^{b\sigma}] (A^c \cdot A_d), \\ \mathcal{L}_5 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\nu}) A^{d\rho} A^{b\sigma}] (A^c \cdot A_d), \\ \mathcal{L}_6 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{d\alpha}) A_\mu^a A_\nu^b A_\rho^c A_\sigma^d], \\ \mathcal{L}_7 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\alpha A^{d\mu}) A_\mu^a A_\nu^b A_\rho^c A_\sigma^d], \\ \mathcal{L}_8 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial \cdot A_d) A^{d\mu} A^{a\nu} A^{b\rho} A^{c\sigma}]. \end{array} \right. \quad (24)$$

Note that one cannot build a single derivative term without an additional field, as it would otherwise belong to the **3** representation of SU(2).

For two first-order vector field derivatives without additional fields, one gets

$$\left\{ \begin{array}{l} \mathcal{L}_1 = (\partial \cdot A^a) (\partial \cdot A_a), \\ \mathcal{L}_2 = [(\partial^\mu A_a^\nu) (\partial_\mu A_\nu^a)], \\ \mathcal{L}_3 = [(\partial^\mu A_a^\nu) (\partial_\nu A_\mu^a)], \\ \mathcal{L}_4 = [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\nu}) (\partial^\rho A_a^\sigma)], \end{array} \right. \quad (25)$$

whereas with two additional vector fields, one gets

$$\left\{ \begin{array}{l} \mathcal{L}_1 = (\partial \cdot A^a) (\partial \cdot A_a) (A^b \cdot A_b), \\ \mathcal{L}_2 = (\partial \cdot A^a) (\partial \cdot A^b) (A_a \cdot A_b), \\ \mathcal{L}_3 = [(\partial^\mu A_a^\nu) (\partial_\mu A_\nu^a)] (A^b \cdot A_b), \\ \mathcal{L}_4 = [(\partial^\mu A_a^\nu) (\partial_\mu A_\nu^b)] (A^a \cdot A_b), \\ \mathcal{L}_5 = [(\partial^\mu A_a^\nu) (\partial_\nu A_\mu^a)] (A^b \cdot A_b), \\ \mathcal{L}_6 = [(\partial^\mu A_a^\nu) (\partial_\nu A_\mu^b)] (A^a \cdot A_b), \\ \mathcal{L}_7 = [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\nu}) (\partial^\rho A_a^\sigma)] (A^b \cdot A_b), \\ \mathcal{L}_8 = [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\nu}) (\partial^\rho A_b^\sigma)] (A_a \cdot A^b), \\ \mathcal{L}_9 = [(\partial^\mu A_a^\nu) A_\mu^a A_\nu^b] (\partial \cdot A_b), \\ \mathcal{L}_{10} = [(\partial^\mu A_a^\nu) A_\mu^b A_\nu^a] (\partial \cdot A_b), \\ \mathcal{L}_{11} = [(\partial^\mu A_a^\nu) A_\mu^b A_{b\nu}] (\partial \cdot A^a), \\ \mathcal{L}_{12} = [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\nu}) A_a^\rho A_b^\sigma] (\partial \cdot A^b), \\ \mathcal{L}_{13} = [A_\mu^a A_{a\nu} (\partial^\mu A_b^\alpha) (\partial^\nu A_\alpha^b)], \\ \mathcal{L}_{14} = [A_\mu^a A_\nu^b (\partial^\mu A_a^\alpha) (\partial^\nu A_{b\alpha})], \\ \mathcal{L}_{15} = [A_\mu^a A_{a\nu} (\partial^\mu A^{b\alpha}) (\partial_\alpha A_\nu^b)], \\ \mathcal{L}_{16} = [A_\mu^a A_\nu^b (\partial^\mu A_a^\alpha) (\partial_\alpha A_\nu^b)], \\ \mathcal{L}_{17} = [A_\mu^a A_\nu^b (\partial^\mu A_b^\alpha) (\partial_\alpha A_\nu^a)], \\ \mathcal{L}_{18} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A^{b\nu} (\partial^\rho A_a^\alpha) (\partial^\sigma A_{b\alpha})], \\ \mathcal{L}_{19} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A^{b\nu} (\partial^\rho A_a^\alpha) (\partial_\alpha A_b^\sigma)], \\ \mathcal{L}_{20} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A^{b\nu} (\partial^\alpha A_a^\rho) (\partial_\alpha A_b^\sigma)], \\ \mathcal{L}_{21} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A_a^\alpha (\partial^\nu A_b^\rho) (\partial^\sigma A_\alpha^b)], \\ \mathcal{L}_{22} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A_b^\alpha (\partial^\nu A_a^\rho) (\partial^\sigma A_\alpha^b)], \\ \mathcal{L}_{23} = [\epsilon_{\mu\nu\rho\sigma} A_a^\mu A_b^\alpha (\partial^\nu A^{b\rho}) (\partial^\sigma A_\alpha^a)], \\ \mathcal{L}_{24} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A_a^\alpha (\partial^\nu A^{b\rho}) (\partial_\alpha A_b^\sigma)], \\ \mathcal{L}_{25} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A_b^\alpha (\partial^\nu A_a^\rho) (\partial_\alpha A^{b\sigma})], \\ \mathcal{L}_{26} = [\epsilon_{\mu\nu\rho\sigma} A_a^\mu A_b^\alpha (\partial^\nu A^{b\rho}) (\partial_\alpha A^{a\sigma})], \\ \mathcal{L}_{27} = [A_\mu^a A_\nu^b (\partial^\mu A_b^\alpha) (\partial^\nu A_{a\alpha})], \\ \mathcal{L}_{28} = [A_\mu^a A_\nu^b (\partial^\alpha A_b^\mu) (\partial_\alpha A_\nu^a)]. \end{array} \right. \quad (26)$$

Finally, with three derivatives, one finds

$$\left\{ \begin{array}{l} \mathcal{L}_1 = \epsilon_{abc} [(\partial^\mu A_\nu^a) (\partial^\nu A_\rho^b) (\partial^\rho A_\mu^c)], \\ \mathcal{L}_2 = \epsilon_{abc} [(\partial^\mu A_\nu^a) (\partial^\nu A_\rho^b) (\partial_\mu A^{c\rho})], \\ \mathcal{L}_3 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\alpha}) (\partial^\nu A_\alpha^b) (\partial^\rho A^{c\sigma})], \\ \mathcal{L}_4 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\mu A^{a\alpha}) (\partial_\alpha A^{b\nu}) (\partial^\rho A^{c\sigma})], \\ \mathcal{L}_5 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} (\partial^\alpha A^{a\mu}) (\partial_\alpha A^{b\nu}) (\partial^\rho A^{c\sigma})], \end{array} \right. \quad (27)$$

completing our list of test Lagrangians.

IV. CONSTRUCTION OF THE HEALTHY TERMS

A. Hessian Condition

Let us now apply the Hessian condition, as discussed in Sec. IID. The first step is to calculate the Hessians associated to the various test Lagrangians, defined through Eq. (11). One sees that only those terms containing at least two first-order derivatives of the vector field yield a non vanishing value. In practice, one gets

$$\left\{ \begin{array}{l} \mathcal{H}_1^{\mu\nu de} = 2g^{0\mu} g^{0\nu} \delta^{de}, \\ \mathcal{H}_2^{\mu\nu de} = -2g^{\mu\nu} \delta^{de}, \\ \mathcal{H}_3^{\mu\nu de} = 2g^{0\mu} g^{0\nu} \delta^{de}, \\ \mathcal{H}_4^{\mu\nu de} = 0, \end{array} \right. \quad (28)$$

for the terms with two first-order derivatives and no additional fields, and

$$\left\{ \begin{array}{l} \mathcal{H}_1^{\mu\nu de} = 2g^{0\mu}g^{0\nu}g^{de}(A^b \cdot A_b), \\ \mathcal{H}_2^{\mu\nu de} = 2g^{0\mu}g^{0\nu}(A^d \cdot A^e), \\ \mathcal{H}_3^{\mu\nu de} = -2g^{\mu\nu}g^{de}(A^b \cdot A_b), \\ \mathcal{H}_4^{\mu\nu de} = -2g^{\mu\nu}(A^d \cdot A^e), \\ \mathcal{H}_5^{\mu\nu de} = 2g^{0\mu}g^{0\nu}g^{de}(A^b \cdot A_b), \\ \mathcal{H}_6^{\mu\nu de} = 2g^{0\mu}g^{0\nu}(A^d \cdot A^e), \\ \mathcal{H}_7^{\mu\nu de} = 0, \\ \mathcal{H}_8^{\mu\nu de} = 0, \\ \mathcal{H}_9^{\mu\nu de} = A^{0d}A^{\mu e}g^{0\nu} + A^{0e}A^{\nu d}g^{0\mu}, \\ \mathcal{H}_{10}^{\mu\nu de} = A^{0e}A^{\mu d}g^{0\nu} + A^{0d}A^{\nu e}g^{0\mu}, \\ \mathcal{H}_{11}^{\mu\nu de} = A^{0b}A_b^\mu g^{0\nu}g^{de} + A^{0b}A_b^\nu g^{0\mu}g^{de}, \\ \mathcal{H}_{12}^{\mu\nu de} = \epsilon^{0\mu}{}_{\rho\sigma}A^{\rho d}A^{\sigma e}g^{0\nu} + \epsilon^{0\nu}{}_{\rho\sigma}A^{\rho e}A^{\sigma d}g^{0\mu}, \\ \mathcal{H}_{13}^{\mu\nu de} = 2A^{0b}A_b^0 g^{\mu\nu}g^{de}, \\ \mathcal{H}_{14}^{\mu\nu de} = 2A^{0d}A^{0e}g^{\mu\nu}, \\ \mathcal{H}_{15}^{\mu\nu de} = A^{0b}A_b^\nu g^{\mu 0}g^{de} + A^{0b}A_b^\mu g^{\nu 0}g^{de}, \\ \mathcal{H}_{16}^{\mu\nu de} = A^{0d}A^{\nu e}g^{\mu 0} + A^{0e}A^{\mu d}g^{\nu 0}, \\ \mathcal{H}_{17}^{\mu\nu de} = A^{0e}A^{\nu d}g^{\mu 0} + A^{0d}A^{\mu e}g^{\nu 0}, \\ \mathcal{H}_{18}^{\mu\nu de} = 0, \\ \mathcal{H}_{19}^{\mu\nu de} = \epsilon_{\rho\sigma}{}^{0\nu}A^{\rho d}A^{\sigma e}g^{\mu 0} + \epsilon_{\rho\sigma}{}^{0\mu}A^{\rho e}A^{\sigma d}g^{\nu 0}, \\ \mathcal{H}_{20}^{\mu\nu de} = -2\epsilon_{\rho\sigma}{}^{\mu\nu}A^{\rho d}A^{\sigma e}, \\ \mathcal{H}_{21}^{\mu\nu de} = 0, \\ \mathcal{H}_{22}^{\mu\nu de} = 0, \\ \mathcal{H}_{23}^{\mu\nu de} = 0, \\ \mathcal{H}_{24}^{\mu\nu de} = 0, \\ \mathcal{H}_{25}^{\mu\nu de} = \epsilon_\rho{}^{0\mu\nu}A^{\rho d}A^{0e} + \epsilon_\rho{}^{0\nu\mu}A^{\rho e}A^{0d}, \\ \mathcal{H}_{26}^{\mu\nu de} = \epsilon_\rho{}^{0\mu\nu}A^{\rho e}A^{0d} + \epsilon_\rho{}^{0\nu\mu}A^{\rho d}A^{0e}, \\ \mathcal{H}_{27}^{\mu\nu de} = 2A^{0d}A^{0e}g^{\mu\nu}, \\ \mathcal{H}_{28}^{\mu\nu de} = -2A^{\nu d}A^{\mu e}, \end{array} \right. \quad (29)$$

for those with two first-order derivatives and two additional vector fields.

For the terms with three first-order derivatives we have

$$\left\{ \begin{array}{l} \mathcal{H}_1^{\mu\nu de} = 3\epsilon_c^{de}(g^{0\mu}\partial^\nu A^{c0} - g^{0\nu}\partial^\mu A^{c0}), \\ \mathcal{H}_2^{\mu\nu de} = \epsilon_c^{de}(g^{0\mu}\partial^0 A^{c\nu} - g^{0\nu}\partial^0 A^{c\mu}) + \epsilon_c^{de}(\partial^\mu A^{c\nu} - \partial^\nu A^{c\mu}), \\ \mathcal{H}_3^{\mu\nu de} = 0, \\ \mathcal{H}_4^{\mu\nu de} = \epsilon_c^{de}(\epsilon^{0\nu\rho\sigma}g^{0\mu}\partial_\rho A_\sigma^c - \epsilon^{0\mu\rho\sigma}g^{0\nu}\partial_\rho A_\sigma^c) + 2\epsilon_c^{de}\epsilon^{\rho\mu 0\nu}\partial_\rho A^{c0}, \\ \mathcal{H}_5^{\mu\nu de} = -2\epsilon_c^{de}\epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma^c + 4\epsilon_c^{de}\epsilon^{\rho\mu 0\nu}\partial^0 A_\rho^c. \end{array} \right. \quad (30)$$

With these partial Hessians, we now construct a basis of terms fulfilling the condition discussed above, i.e. such that $\mathcal{H}^{0\mu de} = 0$ for all values of μ , d and e , see Sec. II D. To reach this goal, using notations already introduced in Ref. [68], we produce a Lagrangian by means of a linear combination of our test ones, namely

$$\mathcal{L}_{\text{test}} = \sum_i x_i \mathcal{L}_i, \quad (31)$$

for a yet-unknown set of constant parameters x_i . The Hessian is then calculated for this Lagrangian, leading to algebraic equations for the x_i whose roots provide the required actions. It turns out to be easier to compute separately the cases $\mu = 0$ and $\mu = i$, as well as $d = e$ and $d \neq e$.

Let us begin with the case $d = e$. Test Lagrangians with two derivatives and no additional fields have only one Hessian component not identically vanishing, namely

$$\mathcal{H}^{00dd} = x_1 + x_2 + x_3 \quad (\text{no sum}), \quad (32)$$

while for two additional vector fields, there are four independent Hessian conditions, given by

$$\begin{aligned}\mathcal{H}^{00dd} &= (x_1 + x_3 + x_5) (A^{\bar{b}} \cdot A_{\bar{b}}) + 2(x_2 + x_4 + x_6) (A^{\bar{d}} \cdot A^{\bar{d}}) \\ &\quad - 2(x_9 + x_{10} + x_{14} + x_{16} + x_{17} + x_{27} + x_{28}) (A^{0d} A^{0d}) \\ &\quad - (x_{11} + x_{13} + x_{15}) (A^{0b} A_b^0),\end{aligned}\tag{33}$$

$$\begin{aligned}\mathcal{H}^{0idd} &= -(x_9 + x_{10} + x_{16} + x_{17} + 2x_{28}) (A^{0d} A^{id}) - \frac{1}{2} (x_{11} + x_{15}) (A^{0b} A_b^i) \\ &\quad - (x_{12} + x_{19} + 2x_{20}) (\epsilon_{\rho\sigma}^{0i} A^{\rho d} A^{\sigma d}).\end{aligned}\tag{34}$$

On the other hand, the case $d \neq e$ implies

$$\mathcal{H}^{00de} = 2(x_2 + x_4 + x_6) (A^{\bar{d}} \cdot A^{\bar{e}}) - 2(x_9 + x_{10} + x_{14} + x_{16} + x_{17} + x_{27} + x_{28}) (A^{0d} A^{0e}),\tag{35}$$

$$\mathcal{H}^{0ide} = -(x_9 + x_{17} + 2x_{28}) (A^{0e} A^{id}) - (x_{10} + x_{16}) (A^{0d} A^{ie}) - (-x_{12} + x_{19} + 2x_{20}) (\epsilon_{\rho\sigma}^{0i} A^{\rho d} A^{\sigma e}).\tag{36}$$

Making these four terms vanish can be done, without lack of generality (since all linear combinations of the resulting terms are all also acceptable):

$$\begin{aligned}x_3 &= -x_1 - x_5, \\ x_4 &= -x_2 - x_6, \\ x_{12} &= 0, \\ x_{13} &= 0, \\ x_{14} &= -x_{27} + x_{28}, \\ x_{15} &= -x_{11}, \\ x_{16} &= -x_{10}, \\ x_{17} &= -x_9 - 2x_{28}, \\ x_{19} &= -2x_{20}.\end{aligned}\tag{37}$$

With three derivatives, one finds that \mathcal{H}^{00dd} , \mathcal{H}^{00de} ($d \neq e$) and \mathcal{H}^{0idd} identically vanish, whereas for $d \neq e$, we have

$$\mathcal{H}^{0ide} = \epsilon_c^{de} [(-3x_1 - x_2) \partial^i A^{0c} - (x_4 + 2x_5) \epsilon^{0i\rho\sigma} \partial_\rho A_\sigma^c],\tag{38}$$

thus leading to the conditions

$$\begin{aligned}x_2 &= -3x_1, \\ x_4 &= -2x_5.\end{aligned}\tag{39}$$

B. Simplification of the Lagrangian

For one gradient and two vector fields, we can define the current

$$J^\mu = \epsilon_{abc} \epsilon^{\mu\nu\rho\sigma} A_\nu^a A_\rho^b A_\sigma^c,\tag{40}$$

showing that \mathcal{L}_2 is a total derivative, namely

$$\partial_\mu J^\mu = 3\mathcal{L}_2.\tag{41}$$

A similar technique applies for one derivative term and 4 additional vector fields: in this case, one forms the following two currents

$$\begin{aligned}J_1^\mu &= \epsilon^\mu{}_{\nu\rho\sigma} A^{\nu a} A^{\rho b} A^{\sigma c} A^{\alpha d} A_{\alpha d} \epsilon_{abc}, \\ J_2^\alpha &= \epsilon_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A^{\rho c} A^{\sigma d} A_d^\alpha \epsilon_{abc},\end{aligned}\tag{42}$$

yielding

$$\begin{aligned}\partial_\mu J_1^\mu &= 3(\mathcal{L}_3 - 2\mathcal{L}_5 + 2\mathcal{L}_6), \\ \partial_\alpha J_2^\alpha &= -\mathcal{L}_8.\end{aligned}\tag{43}$$

Finally, some terms involving two first-order derivatives can be described from

$$J^{\mu_1} = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A^{\nu_1 a} \partial_{\mu_2} A_a^{\nu_2},\tag{44}$$

$$J_\epsilon^\mu = \epsilon^{\mu\nu\rho\sigma} A_\nu^a (\partial_\rho A_{\sigma a}),\tag{45}$$

where we have used the definition $\delta_{\nu_1\nu_2}^{\mu_1\mu_2} \equiv \delta_{\nu_1}^{\mu_1} \delta_{\nu_2}^{\mu_2} - \delta_{\nu_2}^{\mu_1} \delta_{\nu_1}^{\mu_2}$ stemming from Eq. (A2), leading to

$$\partial_{\mu_1} J^{\mu_1} = \mathcal{L}_1 - \mathcal{L}_3,\tag{46}$$

$$\partial_\mu J_\epsilon^\mu = \mathcal{L}_4.\tag{47}$$

Terms containing two derivatives and two fields are slightly more involved. We first make use of the identity [127]

$$A^{\mu\alpha} \tilde{B}_{\nu\alpha} + B^{\mu\alpha} \tilde{A}_{\nu\alpha} = \frac{1}{2} (B^{\alpha\beta} \tilde{A}_{\alpha\beta}) \delta_\nu^\mu,\tag{48}$$

valid for all antisymmetric tensors A and B . This provides the relations

$$(G^{\mu\alpha a} \tilde{G}_{\nu\alpha}^b + G^{\mu\alpha b} \tilde{G}_{\nu\alpha}^a) A_{\mu a} A_b^\nu = \frac{1}{2} (G^{\alpha\beta a} \tilde{G}_{\alpha\beta}^b) (A_a \cdot A_b),\tag{49}$$

and

$$G^{\mu\alpha a} \tilde{G}_{\nu\alpha a} A_\mu^b A_b^\nu = \frac{1}{4} (G^{\alpha\beta a} \tilde{G}_{\alpha\beta a}) (A^b \cdot A_b),\tag{50}$$

where $G^{\mu\alpha a}$ is the Abelian-like Faraday tensor as defined below in Eq. (59). From these, one then derives the following two identities relating the Lagrangians in Eq. (26)

$$\mathcal{L}_{25} + \mathcal{L}_{26} - \mathcal{L}_{22} - \mathcal{L}_{23} = \mathcal{L}_8,\tag{51}$$

and

$$2(\mathcal{L}_{24} - \mathcal{L}_{21}) = \mathcal{L}_7.\tag{52}$$

It is also possible to find total derivatives to reduce the number of independent terms. First, one can use the fact that \tilde{G} is divergence-free, introducing the currents

$$\begin{aligned}J_{G,1}^\mu &= \tilde{G}_a^{\mu\nu} A_\nu^a (A^b \cdot A_b), \\ J_{G,2}^\mu &= \tilde{G}_a^{\mu\nu} A_{\nu b} (A^a \cdot A^b),\end{aligned}\tag{53}$$

providing

$$\begin{aligned}\partial_\mu J_{G,1}^\mu &= \mathcal{L}_7 - 2\mathcal{L}_{22}, \\ \partial_\mu J_{G,2}^\mu &= \mathcal{L}_8 - \mathcal{L}_{21} - \mathcal{L}_{23}.\end{aligned}\tag{54}$$

One can subsequently use the antisymmetric forms written from $\delta_{\nu_1\nu_2}^{\mu_1\mu_2}$:

$$\begin{aligned}J_{\delta,1}^\mu &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b A_a^{\nu_1} \partial_{\mu_2} A_a^{\nu_2}, \\ J_{\delta,2}^\mu &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_\lambda^b A^{\nu_1 a} \partial_{\mu_2} A_b^{\nu_2}, \\ J_{\delta,3}^\mu &= \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A^{\lambda a} A_a^{\nu_1} A_b^{\nu_2} \partial_{\mu_2} A_\lambda^b,\end{aligned}\tag{55}$$

resulting into

$$\begin{aligned}\partial_\mu J_{\delta,1}^\mu &= \mathcal{L}_1 - \mathcal{L}_5 + 2\mathcal{L}_{10} - 2\mathcal{L}_{16}, \\ \partial_\mu J_{\delta,2}^\mu &= \mathcal{L}_2 - \mathcal{L}_6 + \mathcal{L}_9 + \mathcal{L}_{11} - \mathcal{L}_{15} - \mathcal{L}_{17}, \\ \partial_\mu J_{\delta,3}^\mu &= \mathcal{L}_{14} + \mathcal{L}_9 + \mathcal{L}_{15} - \mathcal{L}_{27} - \mathcal{L}_{17} - \mathcal{L}_{11}.\end{aligned}\tag{56}$$

Finally, we can write

$$J_{\epsilon,1}^\mu = \epsilon^\mu{}_{\nu\rho\sigma} A^{\nu a} A^{\rho b} A_a^\alpha \partial^\sigma A_{\alpha b}, \quad (57)$$

implying

$$\partial_\mu J_{\epsilon,1}^\mu = \mathcal{L}_{18} + \mathcal{L}_{23} - \mathcal{L}_{21}. \quad (58)$$

All the above conditions are linearly independent. They allow to write Lagrangians \mathcal{L}_9 , $\mathcal{L}_{10} - \mathcal{L}_{16}$, $\mathcal{L}_{11} - \mathcal{L}_{15}$, \mathcal{L}_{18} , \mathcal{L}_{21} , \mathcal{L}_{22} , \mathcal{L}_{24} , and \mathcal{L}_{25} as functions of the other Lagrangians. Note however that one can always add these to other terms of the final basis for simplification purposes.

Lastly, the current $J^\mu = \epsilon^\mu{}_{\nu\rho\sigma} \partial^\nu A^{\alpha a} \partial^\rho A_\alpha^b A^{\sigma c} \epsilon_{abc}$, permits to simplify one of the terms containing three first-order derivatives by making use of $\partial_\mu J^\mu = \mathcal{L}_3$.

C. A New Basis

One can now rewrite our basis of Lagrangians satisfying the Hessian condition, taking into account the extra relations stemming from the total derivatives and the identity of Ref. [127]. We group our terms to produce a new and more convenient basis, and for that purpose we shall use the Abelian Faraday tensor, namely

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a, \quad (59)$$

as well as its Hodge dual $\tilde{G}_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma a}$, also defined in the usual way. Using the Abelian Faraday tensor to describe a non-Abelian vector field theory may seem a bit weird, but it turns out to simplify considerably our forthcoming considerations since this term naturally appears from the first-order derivatives of the vector field, and cancels in the scalar sector. We shall later move on to a formulation using the actual non-Abelian Faraday tensor, as is given by Eq. (2). Besides, we shall also make use of the symmetric counterpart of the Abelian Faraday tensor, namely

$$S_{\mu\nu}^a = \partial_\mu A_\nu^a + \partial_\nu A_\mu^a. \quad (60)$$

For one first-order derivative of the vector field and two additional vector fields, we obtain

$$\tilde{\mathcal{L}}_1 = 2\mathcal{L}_1 = \epsilon_{abc} [G^{\mu\nu a} A_\mu^b A_\nu^c], \quad (61)$$

and with four additional vector fields, we obtain

$$\begin{cases} \tilde{\mathcal{L}}_1 = 2\mathcal{L}_1 = \epsilon_{abc} [G^{\mu\nu d} A_\mu^a A_\nu^b] (A^c \cdot A_d), \\ \tilde{\mathcal{L}}_2 = \mathcal{L}_2 + \mathcal{L}_3 = \epsilon_{abc} [S^{\mu\nu a} A_\mu^b A_\nu^d] (A^c \cdot A_d), \\ \tilde{\mathcal{L}}_3 = \mathcal{L}_3 - \mathcal{L}_2 = \epsilon_{abc} [G^{\mu\nu a} A_\mu^b A_\nu^d] (A^c \cdot A_d), \\ \tilde{\mathcal{L}}_4 = \mathcal{L}_4 = \epsilon_{abc} [\tilde{G}^{\mu\nu d} A_\mu^a A_\nu^b] (A^c \cdot A_d), \\ \tilde{\mathcal{L}}_5 = \mathcal{L}_5 = \epsilon_{abc} [\tilde{G}^{\mu\nu a} A_\mu^d A_\nu^b] (A^c \cdot A_d), \\ \tilde{\mathcal{L}}_6 = \mathcal{L}_6 - \mathcal{L}_7 = \epsilon_{abc} [\epsilon_{\mu\nu\rho\sigma} G^{\mu\alpha d} A_\alpha^\nu A_\rho^a A_\sigma^b A_\alpha^c]. \end{cases} \quad (62)$$

Terms with two first-order derivatives and no additional fields can be written as

$$\mathcal{L}_1 = 2(\mathcal{L}_2 - \mathcal{L}_3) = G_a^{\mu\nu} G_{\mu\nu}^a, \quad (63)$$

and with two additional fields are given by

$$\begin{cases} \tilde{\mathcal{L}}_1 = \mathcal{L}_1 - \mathcal{L}_5 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b (\partial_{\mu_1} A_{\nu_1}^a) (\partial_{\mu_2} A_{\nu_2}^a), \\ \tilde{\mathcal{L}}_2 = 2(\mathcal{L}_3 - \mathcal{L}_5) = G_a^{\mu\nu} G_{\mu\nu}^a (A^b \cdot A_b), \\ \tilde{\mathcal{L}}_3 = \mathcal{L}_2 - \mathcal{L}_6 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A_{\nu_1}^a) (\partial_{\mu_2} A_{\nu_2}^b), \\ \tilde{\mathcal{L}}_4 = 2(\mathcal{L}_4 - \mathcal{L}_6) = G_a^{\mu\nu} G_{\mu\nu}^a (A^a \cdot A^b), \\ \tilde{\mathcal{L}}_5 = 2\mathcal{L}_7 = \tilde{G}_{\mu\nu}^a G_a^{\mu\nu} (A^b \cdot A_b), \\ \tilde{\mathcal{L}}_6 = 2\mathcal{L}_8 = \tilde{G}_{\mu\nu}^a G_b^{\mu\nu} (A^a \cdot A^b), \\ \tilde{\mathcal{L}}_7 = \mathcal{L}_{18} + \mathcal{L}_{20} - 2\mathcal{L}_{19} = [\epsilon_{\mu\nu\rho\sigma} A^{a\mu} A^{b\nu} G_a^{\rho\alpha} G_{\alpha b}^\sigma], \\ \tilde{\mathcal{L}}_8 = \mathcal{L}_{26} + \mathcal{L}_{23} = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{\alpha b} S^{\alpha\sigma a}, \\ \tilde{\mathcal{L}}_9 = \mathcal{L}_{26} - \mathcal{L}_{23} = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{\alpha b} G^{\alpha\sigma a}, \\ \tilde{\mathcal{L}}_{10} = \mathcal{L}_{14} - \mathcal{L}_{27} = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_{\mu_1}^a A_{\mu_2}^b (\partial^{\mu_1} A_a^\alpha) (\partial^{\mu_2} A_{\alpha b}), \\ \tilde{\mathcal{L}}_{11} = \mathcal{L}_{27} + \mathcal{L}_{28} - 2\mathcal{L}_{17} = A_\mu^a A_\nu^b G_{\alpha b}^\mu G^{\nu\alpha}_a. \end{cases} \quad (64)$$

As anticipated, we obtain 11 independent terms, which correspond to 28 terms to begin with, with 8 constraints and 9 Hessian conditions.

Finally, the three-gradient case yields

$$\begin{cases} \tilde{\mathcal{L}}_1 = 2(\mathcal{L}_1 - 3\mathcal{L}_2) = \epsilon_{abc} G^\mu{}_\nu{}^a G^\nu{}_\rho{}^b G^\rho{}_\mu{}^c, \\ \tilde{\mathcal{L}}_2 = 2\mathcal{L}_4 - \mathcal{L}_3 - \mathcal{L}_5 = \epsilon_{abc} G^{\mu\alpha a} G_\alpha{}^{\nu b} \tilde{G}_{\mu\nu}{}^c. \end{cases} \quad (65)$$

D. Scalar Contribution

Let us now consider the scalar part of the previously developed Lagrangian, as explained in Sec. IID, making the substitution $A_\mu^a \rightarrow \partial_\mu \pi^a$ and writing only those terms that do not identically vanish, using the results of the Appendix, where the useful galileon Lagrangians are provided (Sec. A 2), as well as the linear combinations leading to second-order equations (Sec. A 4).

With one derivative and four vector fields, the only remaining term of the scalar sector out of the original three is

$$\tilde{\mathcal{L}}_2 = \epsilon_{abc} S^{\mu\nu a} A_\mu^b A_\nu^d (A^c \cdot A_d), \quad (66)$$

which does not yield second-order equations in the scalar limit.

Lagrangians involving two derivatives of the vector fields provide

$$\begin{cases} \tilde{\mathcal{L}}_1 = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_b^\lambda A_\lambda^b (\partial_{\mu_1} A_a^{\nu_1}) (\partial_{\mu_2} A^{\nu_2 a}), \\ \tilde{\mathcal{L}}_3 = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}), \\ \tilde{\mathcal{L}}_{10} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_\mu^a A_{\mu_2}^b (\partial^{\mu_1} A_a^\alpha) (\partial^{\mu_2} A_{\alpha b}), \end{cases} \quad (67)$$

leading to the scalar corresponding terms

$$\begin{cases} \tilde{\mathcal{L}}_1|_\pi = \mathcal{L}_{4,\text{I}}^{\text{Gal},3}, \\ \tilde{\mathcal{L}}_3|_\pi = \mathcal{L}_{4,\text{II}}^{\text{Gal},3}, \\ \tilde{\mathcal{L}}_{10}|_\pi = \mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2}. \end{cases} \quad (68)$$

One can derive two linear combinations having second-order equations, namely

$$\tilde{\mathcal{L}}_1|_\pi + 2\tilde{\mathcal{L}}_3|_\pi = \mathcal{L}_{4,\text{I}}^{\text{Gal},3} + 2\mathcal{L}_{4,\text{II}}^{\text{Gal},3}, \quad (69)$$

see Eq. (A33), and

$$\tilde{\mathcal{L}}_{10}|_\pi + \tilde{\mathcal{L}}_3|_\pi = \mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \frac{1}{2} \left(2\mathcal{L}_{4,\text{III}}^{\text{Gal},2} + \mathcal{L}_{4,\text{I}}^{\text{Gal},3} \right) + \frac{1}{2} \left(\mathcal{L}_{4,\text{I}}^{\text{Gal},3} + 2\mathcal{L}_{4,\text{II}}^{\text{Gal},3} \right), \quad (70)$$

yielding second-order equations as each of the three terms of the right hand side of Eq. (70) does so, as shown in the Appendix [see Eqs. (A21), (A31) and (A33)].

E. Final Flat Spacetime Model

Let us regroup the results of the above sections to produce the final theory in the flat spacetime with Minkowskian metric. We first gather most of the new terms induced by the non linear contributions into an arbitrary function $f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a)$. Indeed, this is possible because they not only appear in the systematic procedure we have exposed, but they also satisfy all our conditions; this is equivalent to the general proof discussed in Ref. [70], where the typical term is built out of Levi-Civita tensors, necessarily inducing terms proportional to $\epsilon^{00\dots}$ in the Hessian, and hence vanishing contributions.

Up to now, we have used the Abelian Faraday tensor to express the relevant Lagrangians, although there can be situations in which working with the non-Abelian counterpart in Eq. (2) can be more convenient, in particular in view of the fact that this is the relevant tensor that appears naturally when one extends the theory to its gauged version. This is quite simple since the arbitrary function $f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a)$ can be equivalently written as a new function $\tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a)$ using Eq. (2). It is worth noting that such a change of variable implies no other terms than those already included in the original function.

Gathering the above considerations in a compact form, we obtain a first generic term, reminiscent of the Abelian case, namely

$$\mathcal{L}_2 = f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a) = \tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a). \quad (71)$$

In addition to this term, all the remaining previously derived terms involving contractions with up to six Lorentz indices are

$$\begin{cases} \hat{\mathcal{L}}_1 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_b^\lambda A_\lambda^b (\partial_{\mu_1} A_a^{\nu_1}) (\partial_{\mu_2} A^{\nu_2 a}) + 2\delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}), \\ \hat{\mathcal{L}}_2 = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_a^\lambda A_{\lambda b} (\partial_{\mu_1} A^{\nu_1 a}) (\partial_{\mu_2} A^{\nu_2 b}) + \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_{\mu_1}^a A_{\mu_2}^b (\partial^{\mu_1} A_a^\alpha) (\partial^{\mu_2} A_{\alpha b}), \\ \hat{\mathcal{L}}_3 = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{\alpha b} S^{\alpha\sigma a}, \end{cases} \quad (72)$$

the first two being actually equivalent in the scalar sector, since they lead to the same equations of motion, i.e. those stemming from the galileon Lagrangian containing four scalar fields in the **3** representation of SU(2). Note that there is no term containing only one gradient.

With this general basis, which we shall expand upon in the final discussion section, we can now turn to the covariantization required to apply this category of theories to cosmologically relevant situations.

V. COVARIANTIZATION

A. Procedure

We follow below a procedure similar to that proposed for the galileon case [18, 19, 117], the generalized Proca model [51, 66, 68, 118], and the multigalileon situation [120, 123, 124]. The principle is simple: one first transforms all partial derivatives into covariant ones, and then check that only those terms leading to second-order equations of motion at most are kept.

The pure vector part now contains A and ∇A terms, which translate into A , ∂A , g and ∂g terms. None of these term could lead to any derivative of order higher than two in the equations of motion. On the other hand, the Faraday tensor terms do not yield metric derivatives since partial derivatives can be replaced by covariant ones by virtue of the antisymmetry of these terms. We also leave aside these terms.

As for the scalar part, derivatives of order three or more could appear for curvature. To fix this potential problem, we write the equations of motion in terms of covariant derivatives, and commute them in order to generate the curvature tensor, which contains only second-order derivatives of the metric: the problem stands with the derivatives of the curvature terms. As these particular contributions stem from terms implying at least fourth-order derivatives of the scalar field, it is easy to identify them and to write down the required counterterms.

In practice, this does not show that the resulting equations of motion of the metric do not involve higher-order derivatives of the scalar field. We merely apply the results of Ref. [118], where it was shown that if the equations of motion for the scalar field are safe, then so are those for the metric. This result translates directly to our case.

For many of the terms discussed below, it turns out to be easier to write the Lagrangian as a function of the vector field rather than of its scalar part, even though we are ultimately interested in the latter. Indeed, the scalar Euler-Lagrange equation

$$0 = \frac{\partial \mathcal{L}}{\partial \pi_d} - \nabla_\nu \frac{\partial \mathcal{L}}{\partial (\nabla_\nu \pi_d)} + \nabla_\nu \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \nabla_\nu \pi_d)}, \quad (73)$$

can be written as

$$0 = -\nabla_\nu \frac{\partial \mathcal{L}}{\partial (\nabla_\nu \pi_d)} + \nabla_\nu \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu \nabla_\nu \pi_d)} = -\nabla_\nu \left(\frac{\partial \mathcal{L}}{\partial A_{\nu d}} - \nabla_\mu \frac{\partial \mathcal{L}}{\partial (\nabla_\mu A_{\nu d})} \right). \quad (74)$$

since the action is assumed local in A_μ and therefore cannot contain terms involving non derivative functions of the scalar field π .

In the following sections, we write those terms containing only the curvature and its derivative, or only its derivative, by the respective notation $\mathcal{F}|_R$ or $\mathcal{F}|_{\nabla R}$, where \mathcal{F} is the term whose restriction is being considered. We shall concentrate on terms which are non vanishing in the scalar sector only.

B. Terms in \mathcal{L}^{Gal}

The Lagrangians we have to consider give in the scalar sector

$$\begin{cases} \hat{\mathcal{L}}_1 \Big|_{\pi} = \mathcal{L}_{4,\text{I}}^{\text{Gal},3} + 2\mathcal{L}_{4,\text{II}}^{\text{Gal},3}, \\ \hat{\mathcal{L}}_2 \Big|_{\pi} = \mathcal{L}_{4,\text{II}}^{\text{Gal},3} + \mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2}, \end{cases} \quad (75)$$

where we use the galileon Lagrangians of the Appendix. In the following, working in the vector sector, we will substitute $\partial_{\mu}\pi^a \rightarrow A_{\mu}^a$. Eq. (75) implies that only three independent counterterms will be needed, i.e. those associated with $\mathcal{L}_{4,\text{I}}^{\text{Gal},3}$, $\mathcal{L}_{4,\text{II}}^{\text{Gal},3}$ and $(\mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2})$. We now proceed to finding these counterterms.

First, we have

$$\left\{ \nabla_{\nu} \nabla_{\mu} \left[\frac{\partial \mathcal{L}_{4,\text{I}}^{\text{Gal},3}}{\partial (\nabla_{\mu} A_{\nu d})} \right] \right\} \Big|_R = -2A_b^{\lambda} A_{\lambda}^b R_{\mu\nu} \nabla^{\nu} A^{\mu d} - 2A_b^{\lambda} A_{\lambda}^b A^{\mu d} \nabla^{\nu} R_{\mu\nu}. \quad (76)$$

Introducing

$$\mathcal{L}_{4,\text{I,CT}}^{\text{Gal},3} = \frac{1}{4} A_b^{\lambda} A_{\lambda}^b A_{\mu}^a A_{\mu}^a R, \quad (77)$$

we find that

$$\left\{ \nabla_{\nu} \left[\frac{\partial \mathcal{L}_{4,\text{I,CT}}^{\text{Gal},3}}{\partial (A_{\nu d})} \right] \right\} \Big|_{\nabla R} = A_b^{\lambda} A_{\lambda}^b A^{\mu d} \nabla^{\nu} (g_{\mu\nu} R), \quad (78)$$

which finally implies the equation of motion (EOM)

$$EOM_{\pi} \left(\mathcal{L}_{4,\text{I}}^{\text{Gal},3} + \mathcal{L}_{4,\text{I,CT}}^{\text{Gal},3} \right) \Big|_{\nabla R} = -2A_b^{\lambda} A_{\lambda}^b A^{\mu\alpha} \nabla^{\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0, \quad (79)$$

vanishing by virtue of the properties of the Einstein tensor.

Similarly, for $\mathcal{L}_{4,\text{II}}^{\text{Gal},3}$, we have

$$\left\{ \nabla_{\nu} \nabla_{\mu} \left[\frac{\partial \mathcal{L}_{4,\text{II}}^{\text{Gal},3}}{\partial (\nabla_{\mu} A_{\nu d})} \right] \right\} \Big|_R = -2A_b^{\lambda} A_{\lambda}^d R_{\mu\nu} \nabla^{\nu} A^{\mu b} - 2A_b^{\lambda} A_{\lambda}^d A^{\mu b} \nabla^{\nu} R_{\mu\nu}. \quad (80)$$

Introducing

$$\mathcal{L}_{4,\text{II,CT}}^{\text{Gal},3} = \frac{1}{4} A_b^{\lambda} A_{\lambda a} A^{\mu b} A_{\mu}^a R, \quad (81)$$

which verifies

$$\left\{ \nabla_{\nu} \left[\frac{\partial \mathcal{L}_{4,\text{II,CT}}^{\text{Gal},3}}{\partial (A_{\nu d})} \right] \right\} \Big|_{\nabla R} = A_b^{\lambda} A_{\lambda}^d A^{\mu b} \nabla^{\nu} (g_{\mu\nu} R), \quad (82)$$

we obtain that

$$EOM_{\pi} \left(\mathcal{L}_{4,\text{II}}^{\text{Gal},3} + \mathcal{L}_{4,\text{II,CT}}^{\text{Gal},3} \right) \Big|_{\nabla R} = -2A_b^{\lambda} A_{\lambda}^d A^{\mu b} \nabla^{\nu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0. \quad (83)$$

Finally, using the previous notation

$$\tilde{\mathcal{L}}_{10} = \mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2}, \quad (84)$$

we have

$$\left\{ \nabla_{\nu} \nabla_{\mu} \left[\frac{\partial \tilde{\mathcal{L}}_{10}}{\partial (\nabla_{\mu} A_{\nu d})} \right] \right\} \Big|_R = -2A^{\mu d} A^{\lambda b} R^{\nu}{}_{\rho\lambda\mu} \nabla_{\nu} A_b^{\rho} - 2A^{\mu d} A^{\lambda b} A_b^{\rho} \nabla_{\nu} R^{\nu}{}_{\rho\lambda\mu}. \quad (85)$$

The counterterm to introduce is

$$\mathcal{L}_{10,\text{CT}} = -\frac{1}{2} A^{\mu a} A^{\nu b} A_a^\rho A_b^\sigma R_{\mu\nu\rho\sigma}, \quad (86)$$

giving

$$\nabla_\rho \left(\frac{\partial \mathcal{L}_{10,\text{CT}}}{\partial (A_{\rho d})} \right) \Big|_{\nabla R} = -2 A^{\mu a} A_{\lambda a} A^{\rho d} \nabla^\nu R^\lambda_{\rho\mu\nu} = 2 A^{\mu d} A^{\lambda b} A_b^\rho \nabla_\nu R^\nu_{\rho\lambda\mu}. \quad (87)$$

which gives, as expected

$$EOM_\pi (\tilde{\mathcal{L}}_{10} + \mathcal{L}_{10,\text{CT}}) \Big|_{\nabla R} = 0. \quad (88)$$

Then, to obtain the covariantized form of the action, it is sufficient to add the counterterms obtained in this part to the action given previously in the flat spacetime. The result is summarized in Sec. VI.

C. Coupling with Curvature

Once the derivatives have been covariantized, one must also include possible direct coupling terms between the vector field and the curvature tensors, which we do below in a way entirely similar to that of Ref. [68]. First, we demand contractions with tensors whose divergences vanish on all indices (to ensure integrations by parts to provide no higher-order contributions in the equations of motion) [117, 118]: this means the Einstein tensor as well as

$$L_{\mu\nu\rho\sigma} = 2R_{\mu\nu\rho\sigma} + 2(R_{\mu\sigma}g_{\rho\nu} + R_{\rho\nu}g_{\mu\sigma} - R_{\mu\rho}g_{\nu\sigma} - R_{\nu\sigma}g_{\mu\rho}) + R(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu}), \quad (89)$$

whose symmetries are those of the Riemann tensor, to which it is dual in the sense that it can be written as

$$L^{\alpha\beta\gamma\delta} = -\frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \epsilon^{\gamma\delta\rho\sigma} R_{\mu\nu\rho\sigma}. \quad (90)$$

Even limiting ourselves to the same number of fields as in the flat spacetime situation, many terms are a priori possible. To begin with, all contractions involving a single vector field are impossible. With two such fields, the reasoning is exactly equivalent to the Abelian case, which means the Lagrangians

$$\mathcal{L}_1^{\text{curv}} = G_{\mu\nu} A^{\mu a} A_a^\nu, \quad (91)$$

and

$$\mathcal{L}_2^{\text{curv}} = L_{\mu\nu\rho\sigma} G^{\mu\nu a} G_a^{\rho\sigma}, \quad (92)$$

are acceptable.

Terms in which at least one of the Abelian Faraday tensors is replaced by its Hodge dual can always be rewritten as a contraction between the Riemann tensor and two Faraday tensors, which cannot give second-order equations of motion [118]. One could envisage a contraction with a term like $G^{\mu\rho a} G_a^{\nu\sigma}$, but that is proportional to $\mathcal{L}_2^{\text{curv}}$: to show this, one needs to use the following identity

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon^{\rho\sigma\mu\nu} - \epsilon^{\alpha\rho\sigma\mu} \epsilon^{\beta\gamma\delta\nu} + \epsilon^{\alpha\gamma\delta\nu} \epsilon^{\beta\rho\sigma\mu} + \epsilon^{\alpha\beta\delta\nu} \epsilon^{\rho\gamma\sigma\mu} - \epsilon^{\alpha\beta\gamma\nu} \epsilon^{\rho\delta\sigma\mu} = 0, \quad (93)$$

and the first Bianchi identity.

With three fields, one can obtain a new non vanishing term, in contrast to the Abelian case. This is mostly due to the fact that it is possible to have an antisymmetry in the exchange of two underived vector fields. We get

$$\mathcal{L}_3^{\text{curv}} = L_{\mu\nu\rho\sigma} \epsilon_{abc} G^{\mu\nu a} A^{\rho b} A^{\sigma c}, \quad (94)$$

which is shown to be proportional to $L_{\mu\nu\rho\sigma} \epsilon_{abc} G^{\mu\rho a} A^{\nu b} A^{\sigma c}$, by making use of the previous identity on the Levi-Civita tensor.

Four fields provide, again in contrast to the Abelian situation, the extra contribution

$$\mathcal{L}_4^{\text{curv}} = L_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A_a^\rho A_b^\sigma. \quad (95)$$

It is worth noticing at this point that it is possible to go from the expression of $\mathcal{L}_2^{\text{curv}}$ and $\mathcal{L}_3^{\text{curv}}$ using $G_{\mu\nu}^a$ (the Abelian Faraday tensor) to that using $F_{\mu\nu}^a$ (the non-Abelian one), both of which are equal in an Abelian theory: it is sufficient for this purpose to include the terms $\mathcal{L}_3^{\text{curv}}$ and $\mathcal{L}_4^{\text{curv}}$ only (they are generated by the transformation from $G_{\mu\nu}^a$ to $F_{\mu\nu}^a$).

VI. FINAL MODEL, DISCUSSION

Let us summarize the results obtained for the generalized SU(2) Proca theory. First, we showed that any function of the vector field, Faraday tensor, and its Hodge dual (either in their Abelian or non-Abelian formulation) was possible, i.e.

$$\mathcal{L}_2 = f(A_\mu^a, G_{\mu\nu}^a, \tilde{G}_{\mu\nu}^a) = \tilde{f}(A_\mu^a, F_{\mu\nu}^a, \tilde{F}_{\mu\nu}^a). \quad (96)$$

Such a general \mathcal{L}_2 term involving only gauge-invariant quantities for the derivatives is also present in the Abelian case; we will not discuss it any further since it appears similarly (and for the exact same reasons) in both the Abelian and non-Abelian theories.

Before presenting the other terms contained in the non-Abelian action, let us pursue the summary of what had been found for its Abelian counterpart, as worked out in Refs. [66, 68–70]; we denote as usual \mathcal{L}_{n+2} the Lagrangians containing $n \geq 1$ first order derivatives of the vector field. First, the relation between the more general scalar and vector theories, i.e. the galileon and generalized Proca models, provides in this case a deeper understanding through the use of the Stückelberg trick to go from one sector to another (i.e. switching between $\partial_\mu \pi$ and A_μ). In the scalar galileon theory, only one term exists in the Lagrangians \mathcal{L}_3 to \mathcal{L}_5 , each of which generates a contribution to the vector sector by the Stückelberg trick, i.e. those with a pre-factor $f_i(X)$ in the conclusion of Ref. [70]. An additional freedom stems from the fact that a given scalar Lagrangian can give different vector Lagrangians when permuting the second order derivatives before introducing the vector field: although $\partial_\mu \partial_\nu \pi = \partial_\nu \partial_\mu \pi$, this symmetry is absent in the pure vector case since $\partial_\mu A_\nu \neq \partial_\nu A_\mu$. This property led to one additional contribution to the vector sector of each \mathcal{L}_4 to \mathcal{L}_6 . These contributions appear with the pre-factor $g_i(X)$ in Ref. [70]; they vanish in the pure scalar sector.

Coming back to the non-Abelian situation, and in addition to \mathcal{L}_2 , we derived those relevant Lagrangians implying up to 6 contracted Lorentz indices and being non trivial in flat spacetime. Contrary to the Abelian case, we found no such Lagrangian for $n = 1$. For $n = 2$, there are three possible terms, i.e. \mathcal{L}_4 contains

$$\left\{ \begin{array}{l} \mathcal{L}_4^1 = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_b^\lambda A_\lambda^b (\nabla_{\mu_1} A_{\nu_1}^a) (\nabla_{\mu_2} A_{\nu_2}^a) + \frac{1}{4} A_b^\lambda A_\lambda^b A_\mu^a A_\mu^a R \\ \quad + 2 \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_a^\lambda A_{\lambda b} (\nabla_{\mu_1} A_{\nu_1}^a) (\nabla_{\mu_2} A_{\nu_2}^b) + \frac{1}{2} A_b^\lambda A_{\lambda a} A^{\mu b} A_\mu^a R, \\ \mathcal{L}_4^2 = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_a^\lambda A_{\lambda b} (\nabla_{\mu_1} A_{\nu_1}^a) (\nabla_{\mu_2} A_{\nu_2}^b) + \frac{1}{4} A_b^\lambda A_{\lambda a} A^{\mu b} A_\mu^a R \\ \quad + \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} A_{\mu_1}^a A_{\mu_2}^b (\nabla^{\nu_1} A_a^\alpha) (\nabla^{\nu_2} A_{\alpha b}) - \frac{1}{2} A^{\mu a} A^{\nu b} A_a^\rho A_b^\sigma R_{\mu\nu\rho\sigma}, \\ \mathcal{L}_4^3 = \tilde{G}_{\mu\sigma}^b A_a^\mu A_{\alpha b} S^{\alpha\sigma a}, \end{array} \right. \quad (97)$$

the first two terms giving, once developed, the following forms

$$\left\{ \begin{array}{l} \mathcal{L}_4^1 = (A_b \cdot A^b) [(\nabla \cdot A_a) (\nabla \cdot A^a) - (\nabla_\mu A_a^\nu) (\nabla^\mu A_\nu^a) + \frac{1}{4} A_a \cdot A^a R] \\ \quad + 2(A_a \cdot A_b) [(\nabla \cdot A^a) (\nabla \cdot A^b) - (\nabla_\mu A^{\nu a}) (\nabla^\mu A_\nu^b) + \frac{1}{2} A^a \cdot A^b R], \\ \mathcal{L}_4^2 = (A_a \cdot A_b) [(\nabla \cdot A^a) (\nabla \cdot A^b) - (\nabla_\mu A^{\nu a}) (\nabla^\mu A_\nu^b) + \frac{1}{4} A^a \cdot A^b R] \\ \quad + (A^{\mu a} A^{\nu b}) [(\nabla_\mu A_a^\alpha) (\nabla_\nu A_{\alpha b}) - (\nabla_\nu A_a^\alpha) (\nabla_\mu A_{\alpha b}) - \frac{1}{2} A_b^\rho A^{\sigma b} R_{\mu\nu\rho\sigma}], \end{array} \right. \quad (98)$$

which are more easily compared with the equivalent results for the abelian case. Finally, we also found four extra possibilities for the Lagrangians implying a coupling with the curvature

$$\begin{aligned} \mathcal{L}_1^{\text{curv}} &= G_{\mu\nu} A^{\mu a} A_\nu^a, \\ \mathcal{L}_2^{\text{curv}} &= L_{\mu\nu\rho\sigma} F_a^{\mu\nu} F_{\mu\nu}^a, \\ \mathcal{L}_3^{\text{curv}} &= L_{\mu\nu\rho\sigma} \epsilon_{abc} F^{\mu\nu a} A^{\rho b} A^{\sigma c}, \\ \mathcal{L}_4^{\text{curv}} &= L_{\mu\nu\rho\sigma} A^{\mu a} A^{\nu b} A_a^\rho A_b^\sigma, \end{aligned} \quad (99)$$

thereby completing the full action at that order.

Let us first consider the actions whose equations of motion involve only second-order derivatives for the scalar (not first-order), which is equivalent to having only two vector fields together with the relevant gradients in the action. The multi-galileon SU(2) model in the adjoint representation has been considered in [100], where it was showed that building a Lagrangian is only possible at the order of \mathcal{L}_4 (not to mention the order \mathcal{L}_2 already discussed above). The equivalent formulations of this Lagrangian are detailed in the Appendix A. Following the previous considerations, no Lagrangian in the vector sector should appear at the order of \mathcal{L}_3 since there is no such associated Lagrangian for the multi-galileon at that order; we explicitly confirmed this expectation. In addition, two Lagrangians should appear at of the order of \mathcal{L}_4 ,

one associated to the multi-galileon dynamics, and one associated to the commutation of second order derivatives of the scalar field. In fact, three Lagrangians have been found, two of them giving the multi-galileon dynamics in the scalar sector. We then interpret these two previous terms as contributions which are equivalent in the scalar case, but not in the vector case. It is also due to a commutation of the second order derivatives of the scalar fields, but in a current term, which implies that it is not possible to describe this commutation with a Lagrangian vanishing in the pure scalar sector. This additional term is specific to the non abelian case: the term in $\delta_{\nu_1\nu_2}^{\mu_1\mu_2} A_{\mu_1}^a A_{\mu_2}^b (\nabla^{\nu_1} A_a^\alpha) (\nabla^{\nu_2} A_{\alpha b})$ vanishes in the abelian case, while \mathcal{L}_4^1 and \mathcal{L}_4^2 both reduce to $\mathcal{L}_4^{\text{abelian}} = \delta_{\nu_1\nu_2}^{\mu_1\mu_2} A^\lambda A_\lambda (\nabla_{\mu_1} A^{\nu_1}) (\nabla_{\mu_2} A^{\nu_2})$.

To go further, let us first consider terms implying more derivatives, i.e. having $n \geq 3$. At the order of \mathcal{L}_5 , and since there is no possible dynamics for the SU(2) adjoint multi-galileon, we expect no term having a non-vanishing pure scalar contribution to be possible. This suggests that the only possible term is

$$\mathcal{L}_5 = \epsilon_{abc} (A^a \cdot A^d) \tilde{G}_d^{\alpha\mu} \tilde{G}_\mu^{\beta b} S_{\alpha\beta}^c, \quad (100)$$

the other SU(2) index contractions giving a vanishing result. At the order of \mathcal{L}_6 , the only possibility seems to be the independent possible contractions of SU(2) indices on $\mathcal{L}_6^{\text{abelian}} = (A \cdot A) \tilde{G}^{\alpha\beta} \tilde{G}^{\mu\nu} S_{\alpha\mu} S_{\beta\nu}$, since there is no possibility to have a term which does not vanish in the pure scalar sector. However, one should verify that there is no other term vanishing in the pure scalar sector, not included in \mathcal{L}_2 , and whose dynamics is not described by the previous ones. This kind of terms would be specific to a non abelian theory, as is the second term of \mathcal{L}_4^2 , and vanish for a vector field in a trivial group representation.

Concerning the Lagrangians with more than two vector fields together with the relevant gradients, one has to pay attention to the fact that fully factorizing a $f(A_\mu^a)$ as in the abelian case term is not guaranteed to lead to a valid procedure, although factorizing such an arbitrary function in front of any valid contribution also leads to another valid contribution. Besides, one could think that if there exists no valid Lagrangian with only a few non gradient vector fields at a given derivative order, it is fairly probable that there also exists no such valid Lagrangian at all at this order. For instance, we showed explicitly that terms at the order of \mathcal{L}_3 are not possible with up to 4 vector fields, and this questions the possibility to have such a term even with a higher number of vector fields. An interesting point is that if a Lagrangian is allowed which does not vanish in the pure scalar sector, it corresponds to a possible term in the multi-galileon action, which shows that both theories are closely related.

To conclude, this discussion showed that even if the full action of the model has not been obtained yet, discussing the low order terms permits to identify and understand the whole Lagrangian structure. The above discussion is not specific to the SU(2) case, and therefore can be extended to other group representations. For a theory with a vector field transforming under any representation of any group, a systematic study of all possible terms in the action should be performed in parallel with the corresponding multi-galileon theory.

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Appendix A: SU(2) Galileon Lagrangians Equivalent Formulations

1. Introduction

The purpose of this appendix is to write explicitly all the Lagrangians describing the multi-galileon dynamics in the 3-dimensional representation of SU(2), focusing on the Lagrangians containing only four galileon fields, i.e. those which are useful in this article. A Lagrangian describing this dynamics is given in Ref. [123], namely

$$\mathcal{L}_m^\pi = \alpha^{i_1 \dots i_m} \delta_{[\nu_2 \dots \nu_m]}^{\mu_2 \dots \mu_m} \pi_{i_1} \partial_{\mu_2} \partial^{\nu_2} \pi_{i_2} \dots \partial_{\mu_m} \partial^{\nu_m} \pi_{i_m}, \quad (A1)$$

with m running from 1 to 5, and with the notation

$$\frac{1}{(D-n)!} \epsilon^{i_1 \dots i_n \sigma_1 \dots \sigma_{D-n}} \epsilon_{j_1 \dots j_n \sigma_1 \dots \sigma_{D-n}} = n! \delta_{[i_1 \dots i_n]}^{j_1 \dots j_n} = \delta_{i_1 \dots i_n}^{j_1 \dots j_n} = \delta_{i_1}^{j_1} \dots \delta_{i_n}^{j_n} \pm \dots, \quad (A2)$$

for n running from 1 to 4 (in a four-dimensional spacetime). Other equivalent formulations are possible, whose writing is the purpose of this appendix.

This investigation is necessary for two reasons. First, the formulation given in Eq. (A1) cannot be obtained from a vector Lagrangian using the switch $A_\mu^a \rightarrow \partial_\mu \pi^a$, since a scalar field without derivatives is present. Second, if different Lagrangians are equivalent in the scalar sector, they could give not equivalent Lagrangians in the vector sector. We can thus expect that different Lagrangians valid in the vector sector become different but equivalent formulations of the multi-galileon dynamics when considering the pure scalar part of the action.

For this purpose, we will use the results of Ref. [16], which describe equivalent formulations of the galileon theory in the Abelian case, introducing a Lagrangian similar to that in Eq. (A1) together with the following Lagrangians:

$$\mathcal{L}_m^{\text{Gal},1} = \delta_{[\nu_1 \dots \nu_{m-1}]^{\mu_1 \dots \mu_{m-1}}} \partial_{\mu_1} \pi \partial^{\nu_1} \pi \partial_{\mu_2} \partial^{\nu_2} \pi \dots \partial_{\mu_{m-1}} \partial^{\nu_{m-1}} \pi, \quad (\text{A3})$$

$$\mathcal{L}_m^{\text{Gal},2} = \delta_{[\nu_1 \dots \nu_{m-2}]^{\mu_1 \dots \mu_{m-2}}} \partial_{\mu_1} \pi \partial_\lambda \pi \partial^{\nu_1} \partial^\lambda \pi \dots \partial_{\mu_{m-2}} \partial^{\nu_{m-2}} \pi, \quad (\text{A4})$$

$$\mathcal{L}_m^{\text{Gal},3} = \delta_{[\nu_1 \dots \nu_{m-2}]^{\mu_1 \dots \mu_{m-2}}} \partial_\lambda \pi \partial^\lambda \pi \partial_{\mu_1} \partial^{\nu_1} \pi \dots \partial_{\mu_{m-2}} \partial^{\nu_{m-2}} \pi, \quad (\text{A5})$$

for $m \geq 2$, the case $m = 1$ giving $\mathcal{L} = \pi$. They all give second-order equations of motion.

2. Lagrangians

We first write all the possible Lagrangians appearing when we add the group indices to the previous Lagrangians, restricting ourselves to the case $m = 4$. They are more numerous than in the multi-galileon case since we have an additional freedom when choosing the group index contractions.

The only possible Lagrangian associated to the formulation of Ref. [123] is

$$\mathcal{L}_4^{\text{PSZ}} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \pi_a \partial_{\mu_1} \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b. \quad (\text{A6})$$

The Lagrangians appearing in Ref. [16], given in Eqs. (A3) to (A5), can be endowed with SU(2) indices in several ways, namely two possibilities for $\mathcal{L}_4^{\text{Gal},1}$:

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \partial_{\mu_1} \pi_a \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b, \quad (\text{A7})$$

and

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \partial_{\mu_1} \pi_a \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^a \partial_{\mu_3} \partial^{\nu_3} \pi^b, \quad (\text{A8})$$

three possibilities for $\mathcal{L}_4^{\text{Gal},2}$:

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},2} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \pi_a \partial_\lambda \pi^a \partial^\lambda \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A9})$$

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},2} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \pi_a \partial_\lambda \pi_b \partial^\lambda \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A10})$$

and

$$\mathcal{L}_{4,\text{III}}^{\text{Gal},2} = \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \pi_a \partial_\lambda \pi_b \partial^\lambda \partial^{\nu_1} \pi^b \partial_{\mu_2} \partial^{\nu_2} \pi^a, \quad (\text{A11})$$

and finally two possibilities for $\mathcal{L}_4^{\text{Gal},3}$:

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},3} = \partial_\lambda \pi_a \partial^\lambda \pi^a \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A12})$$

and

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},3} = \partial_\lambda \pi_a \partial^\lambda \pi_b \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial_{\mu_1} \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi^b. \quad (\text{A13})$$

Looking for the Lagrangians implying second-order equations of motion, one can quickly verify that $\mathcal{L}_4^{\text{PSZ}}$, $\mathcal{L}_{4,\text{I}}^{\text{Gal},1}$ and $\mathcal{L}_{4,\text{II}}^{\text{Gal},1}$ have this property due to the symmetry properties of $\delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3}$. However, the other Lagrangians do not give a priori second-order equations of motion². We then investigate in the following the relations among the different Lagrangians.

² The automatic cancellation between third-order derivatives discussed in Ref. [16] is not valid anymore since this cancellation can be spoiled by the group indices.

3. Relations among the Lagrangians

a. Between PSZ and Gal,1

We first relate $\mathcal{L}_4^{\text{PSZ}}$ and the Lagrangians $\mathcal{L}_4^{\text{Gal},1}$ by means of conserved currents. Indeed,

$$J_{0,\text{I}}^{\mu_1} = J_{4,\text{I}}^{\text{PSZ-Gal},\mu_1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \pi_a \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b, \quad (\text{A14})$$

gives

$$\partial_{\mu_1} J_{0,\text{I}}^{\mu_1} = \partial_{\mu_1} J_{4,\text{I}}^{\text{PSZ-Gal},\mu_1} = \mathcal{L}_4^{\text{PSZ}} + \mathcal{L}_{4,\text{I}}^{\text{Gal},1}, \quad (\text{A15})$$

and

$$J_{0,\text{II}}^{\mu_1} = J_{4,\text{II}}^{\text{PSZ-Gal},\mu_1} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \pi_a \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^a \partial_{\mu_3} \partial^{\nu_3} \pi^b, \quad (\text{A16})$$

gives

$$\partial_{\mu_1} J_{0,\text{II}}^{\mu_1} = \partial_{\mu_1} J_{4,\text{II}}^{\text{PSZ-Gal},\mu_1} = \mathcal{L}_4^{\text{PSZ}} + \mathcal{L}_{4,\text{II}}^{\text{Gal},1}. \quad (\text{A17})$$

It is also possible to make a direct correspondence between $\mathcal{L}_{4,\text{I}}^{\text{Gal},1}$ and $\mathcal{L}_{4,\text{II}}^{\text{Gal},1}$ with the current

$$J_{0,\text{I} \rightarrow \text{II}}^{\mu_2} = J_{4,\text{I} \rightarrow \text{II}}^{\text{Gal},1,\mu_2} = \delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} \partial_{\mu_1} \pi_a \partial^{\nu_1} \pi^a \partial^{\nu_2} \pi_b \partial_{\mu_3} \partial^{\nu_3} \pi^b, \quad (\text{A18})$$

yielding

$$\partial_{\mu_2} J_{0,\text{I} \rightarrow \text{II}}^{\mu_2} = \partial_{\mu_2} J_{4,\text{I} \rightarrow \text{II}}^{\text{Gal},1,\mu_2} = \mathcal{L}_{4,\text{I}}^{\text{Gal},1} - \mathcal{L}_{4,\text{II}}^{\text{Gal},1}. \quad (\text{A19})$$

b. Between Gal,2 and Gal,3

Introducing

$$J_1^{\mu_1} = J_{4,\text{I}}^{\text{Gal},2-3,\mu_1} = \partial_\lambda \pi_a \partial^\lambda \pi^a \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial^{\nu_1} \pi_b \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A20})$$

we get

$$\partial_{\mu_1} J_1^{\mu_1} = \partial_{\mu_1} J_{4,\text{I}}^{\text{Gal},2-3,\mu_1} = 2\mathcal{L}_{4,\text{III}}^{\text{Gal},2} + \mathcal{L}_{4,\text{I}}^{\text{Gal},3}. \quad (\text{A21})$$

In a similar way

$$J_2^{\mu_1} = J_{4,\text{II}}^{\text{Gal},2-3,\mu_1} = \partial_\lambda \pi_a \partial^\lambda \pi_b \delta_{\nu_1 \nu_2}^{\mu_1 \mu_2} \partial^{\nu_1} \pi^a \partial_{\mu_2} \partial^{\nu_2} \pi^b, \quad (\text{A22})$$

we obtain

$$\partial_{\mu_1} J_2^{\mu_1} = \partial_{\mu_1} J_{4,\text{II}}^{\text{Gal},2-3,\mu_1} = \mathcal{L}_{4,\text{I}}^{\text{Gal},2} + \mathcal{L}_{4,\text{II}}^{\text{Gal},2} + \mathcal{L}_{4,\text{II}}^{\text{Gal},3}. \quad (\text{A23})$$

c. Between Gal,1, Gal,2 and Gal,3 through Kronecker properties

We use the following identity given in Ref. [16]:

$$\delta_{\nu_1 \dots \nu_n}^{\mu_1 \dots \mu_n} = \delta_{\nu_1}^{\mu_1} \delta_{\nu_2 \dots \nu_n}^{\mu_2 \dots \mu_n} + \sum_{i=2}^n (-1)^{i-1} \delta_{\nu_i}^{\mu_i} \delta_{\nu_1 \nu_2 \dots \nu_{i-1} \nu_{i+1} \dots \nu_n}^{\mu_1 \mu_2 \dots \mu_n}, \quad (\text{A24})$$

which gives for $n = 3$,

$$\delta_{\nu_1 \dots \nu_3}^{\mu_1 \dots \mu_3} = \delta_{\nu_1}^{\mu_1} \delta_{\nu_2 \nu_3}^{\mu_2 \mu_3} - \delta_{\nu_2}^{\mu_1} \delta_{\nu_1 \nu_3}^{\mu_2 \mu_3} + \delta_{\nu_3}^{\mu_1} \delta_{\nu_1 \nu_2}^{\mu_2 \mu_3}. \quad (\text{A25})$$

It is then possible to obtain two additional relations among the different Lagrangians. Indeed, applying this identity to $\mathcal{L}_{4,\text{I}}^{\text{Gal},1}$ and $\mathcal{L}_{4,\text{II}}^{\text{Gal},1}$, we get

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},1} = -2\mathcal{L}_{4,\text{I}}^{\text{Gal},2} + \mathcal{L}_{4,\text{I}}^{\text{Gal},3}, \quad (\text{A26})$$

and

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},1} = -\mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2} + \mathcal{L}_{4,\text{II}}^{\text{Gal},3}. \quad (\text{A27})$$

4. Lagrangians with Second-Order Equations of Motion

Using the results of the previous subsections, we can summarize the Lagrangians which give second-order equations of motion:

$$\mathcal{L}_4^{\text{PSZ}}, \quad (\text{A28})$$

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},1} = -2\mathcal{L}_{4,\text{I}}^{\text{Gal},2} + \mathcal{L}_{4,\text{I}}^{\text{Gal},3} = -\mathcal{L}_4^{\text{PSZ}} - \partial_\mu J_{0,\text{I}}^\mu, \quad (\text{A29})$$

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},1} = -\mathcal{L}_{4,\text{II}}^{\text{Gal},2} - \mathcal{L}_{4,\text{III}}^{\text{Gal},2} + \mathcal{L}_{4,\text{II}}^{\text{Gal},3} = -\mathcal{L}_4^{\text{PSZ}} - \partial_\mu J_{0,\text{II}}^\mu, \quad (\text{A30})$$

$$\mathcal{L}_{4,\text{II}}^{\text{Gal},2} = \frac{1}{4}\mathcal{L}_{4,\text{I}}^{\text{Gal},1} - \frac{1}{2}\mathcal{L}_{4,\text{II}}^{\text{Gal},1} - \frac{1}{4}\partial_\mu J_1^\mu + \frac{1}{2}\partial_\mu J_2^\mu, \quad (\text{A31})$$

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},2} + \mathcal{L}_{4,\text{III}}^{\text{Gal},2} = -\frac{1}{2}\mathcal{L}_{4,\text{I}}^{\text{Gal},1} + \frac{1}{2}\partial_\mu J_1^\mu, \quad (\text{A32})$$

and

$$\mathcal{L}_{4,\text{I}}^{\text{Gal},3} + 2\mathcal{L}_{4,\text{II}}^{\text{Gal},3} = \frac{1}{2}\mathcal{L}_{4,\text{I}}^{\text{Gal},1} + \mathcal{L}_{4,\text{II}}^{\text{Gal},1} + \frac{1}{2}\partial_\mu J_1^\mu + \partial_\mu J_2^\mu. \quad (\text{A33})$$

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